Line spreads of $PG(5, 2)$

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Abstract

All line spreads of $PG(5, 2)$ are constructed and classified up to equivalence by exhaustive generation considering the specific properties of the automorphism group, and the participation of the spread lines in the subspaces of dimension 3. There are 131044 inequivalent spreads. The orders of the automorphism groups preserving the spreads, and the 2-ranks of the related by Rahilly's construction affine 2-(64,16,5) designs are also computed.

Keywords: Projective space; classification; line spread; automorphism; combinatorial design;

1 Introduction

For the basic concepts and notations concerning combinatorial designs and projective spaces, refer, for instance, to [1], [2], [3], [5], [6], or [15].

1.1 Line spreads

A *t- spread* in $PG(d, q)$ is a set S of t-dimensional subspaces such that any point of the geometry is on exactly one element of S. The subject of this paper are the 1-spreads of $PG(d, q)$ whose elements are disjoint lines, and we shall call them line spreads or just spreads.

A partial spread in $PG(d, q)$ is a set of lines, no pair of which has a point in common. A partial spread in $PG(d, q)$ is maximal if it is not properly contained in any partial spread of $PG(d, q)$. In this context a partial spread in $PG(d, q)$ which forms a partition of the points, is called a spread.

Two partial spreads in $PG(d, q)$ are equivalent if there is an automorphism mapping one to the other.

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For two subspaces P_1 and P_2 of $PG(d,q)$, denote by $\langle P_1, P_2\rangle$ the subspace of smallest dimension containing them. A line spread S in $PG(d,q)$ is *regular (geometric)* if for any three lines $A, B, C \in S$ either $C \in \langle A, B\rangle$ or C contains no points in $\langle A, B\rangle$.

A line in a combinatorial design through a pair of points x, y is the intersection of all blocks containing x and y. A line spread of the design is a partition of the point set of a design into disjoint lines.

1.2 Rahilly's construction [12]

If D is a symmetric $2-(16m-1, 8m-1, 4m-1)$ design whose dual design D^* admits a spread S of lines of size 3, we can define an affine $2-(16m, 4m, (4m-1)/3)$ design d (below we will call such designs $R-derived$) as follows: The point set of d consists of the points of D plus one new point w. Let B_1, B_2, B_3 be three blocks of D that correspond to a line in D^* from the spread S. Let $M = B_1 \cap B_2 \cap B_3$. Define a parallel class C of d consisting of the four blocks $B'_1 = B_1 - M$, $B'_2 = B_2 - M$, $B'_3 = B_3 - M$, $B'_4 = M \cup \{w\}$.

1.3 $PG(5,2)$ - automorphism group, subspaces and related designs

There are 63 points and 651 lines in $PG(5,2)$. A g-dimensional subspace ($g = 0, 1, 2, 3, 4$) has $2^{g+1} - 1$ points and $(2^{2g+1} + 1)/3 - 2^g$ lines. The lines form an $STS(63)$, and the hyperplanes a Hadamard 2-(63,31,15) design.

Any two disjoint lines of $PG(5,2)$ determine a 3-dimensional subspace.

Affine R-derived 2-(64,16,5) designs can be constructed from the spreads of $PG(5,2)$.

The full automorphism group G of $PG(5,2)$ is isomorphic to the projective general linear group $PGL(6,2)$. The group is doubly transitive both on the points and on the lines. An automorphism fixes either no points, or all the points of a subspace. For each subspace P there is a subgroup fixing its points, and this subgroup is transitive on the points outside P. Since a subspace has $1, 3, 7, 15$, or 31 points, the order of the full automorphism group is

$|PGL(6,2)| = 63(63-1)(63-3)(63-7)(63-15)(63-31) = 20158709760.$

1.4 The rank of geometric designs

Hamada formulated a conjecture [8] according to which the design obtained from the points and subspaces of a given dimension in the affine geometry $AG(n, p^m)$ (p a prime) has minimal p-rank, and all the other designs with the same parameters have greater p-rank. Counterexamples to this conjecture are given by five 3-(32,8,7) designs [14] and three 2-(64,16,5) designs [9], [11]. The latter are of rank 16. Two of them can be obtained as R-derived designs of the point-hyperplane design in PG(5,2). One of these two is isomorphic to the point-hyperplane design in $AG(3,4)$. Further investigations on 2-(64,16,5) designs of minimum rank are made in [10] on the R-derived of the Hadamard 2-(63,31,15) designs invariant under the dihedral group of order 10. No new exceptions were found.

1.5 Known classifications of spreads in $PG(d,q)$

Soicher [13] classified partial spreads of lines in finite projective spaces using the GRAPE package within the GAP system. His results include the construction up to equivalence of all partial spreads in $PG(3,2)$, $PG(4,2)$ and $PG(3,3)$, all maximal partial spreads in $PG(3,4)$ and the maximal partial spreads in $PG(3,7)$ of size 45 and invariant under a group of order 5. Then Blokhuis, Brouwer and Wilbrink classified all maximal partial spreads of size 45 in $PG(3,7)$ [4].

Looking for affine 2-(64,16,5) designs of small rank Mavron, McDonough and Tonchev constructed [11] more than 30000 spreads of $PG(5,2)$ and classified them with respect to the rank of the R-derived 2-(64,16,5) designs.

1.6 The present work

The problem of classifying spreads is closely related to the problems of classifying other combinatorial objects, such as design resolutions, designs, codes, graphs, etc. The technique of generating lexicographically minimal representatives is well developed and a very good summary of methods and results can be found in [7]. We have to remark here that most of these results were only possible after a careful consideration of some specific properties of the classified combinatorial objects.

We use the same general approach. The difficulty of this particular problem is in the vast number of equivalent solutions and the practical impossibility to prove the inequivalence of two spreads by checking that they cannot be transformed into one another by anyone of the 20158709760 automorphisms of $PG(5,2)$. We show that the automorphism group allows us to fix three of the spread elements and we choose a numbering of the points which is most convenient for this fixing. We next prove that since any two of the constructed spreads have these three same elements, they can be transformed into each other by at most 1451520 automorphisms. Finding and applying only these automorphisms makes the computation possible. In the parallel calculations for some subcases, both the fixed elements can be more and the automorphisms to try less, and in addition we also use the participation of the spread lines in the 3-dimensional subspaces to make the equivalence check faster.

This work is a natural successor of [10], in which line spreads of Hadamard 2-(63,31,15) designs, which correspond to minimum rank 2-(64,16,5) designs are constructed. In the present work we again use design approach to the problem. We actually make all the computations on the related to $PG(5,2)$ designs, namely, we construct the spreads from blocks of the $STS(63)$, and we compute the needed automorphism groups as automorphism groups of the Hadamard 2-(63,31,15) design. Thus we construct all 131044 inequivalent spreads, and present results about the number of spread lines in the three-dimensional subspaces, the rank of the R-derived 2-(64,16,5) designs, and the automorphism groups, which preserve the spreads.

Our results show that although not all the spreads were constructed in [11], representatives of R-designs of all possible ranks were found, and there are no more minimal rank R-designs except the two ones presented in that paper.

2 Construction of all inequivalent spreads

Denote a line through the points a, b and c by $\{a, b, c\}$. Without loss of generality we can denote the points of $PG(5,2)$ by the numbers $1, 2, ..., 63$ in such a way that $\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, ..., \{61,62,63\}$ is a spread, and the point set $\{1,2,3,...,15\}$ forms a 3-dimensional subspace. All the information about the subspaces and automorphisms can be obtained from the related point-hyperplane 2-(63,31,15) design. That is why we present it in Figure 1. Since a line is defined by two of its points, we will next write $\langle a, b\rangle$ (or $\langle a, c\rangle$ or $\langle b, c\rangle$) instead of $\{a, b, c\}$.

Any two disjoint lines A and B determine a 3-dimensional subspace $\langle A, B\rangle$. In a regular spread the number of spread lines in a 3-dimensional subspace is either 5, or 1. In a spread that is not regular, this number can vary from 0 to 5.

G is doubly transitive on the lines, and thus each pair of spread lines can be mapped by some automorphism into the pair $\langle 1,2\rangle$ and $\langle 4,5\rangle$. Respectively each 3-dimensional subspace can be mapped into the space $P_1 = \langle\langle 1,2\rangle, \langle 4,5\rangle\rangle$.

For each line X, which has no point in P_1, there is an automorphism fixing the points of P_1, and mapping X into $\langle 16,17\rangle$. Therefore for each 3-dimensional subspace P containing points 1 and 2, and different from P_1, there is an automorphism fixing the points of P_1, and mapping P into the 3-dimensional subspace $P_2 = \langle\langle 1,2\rangle, \langle 16,17\rangle\rangle$.

Each spread has lines containing no points in P_1, and therefore there exists an automorphism, mapping it into a spread containing lines $\langle 1,2\rangle$, $\langle 4,5\rangle$, and $\langle 16,17\rangle$. That is why we only construct spreads containing these three lines.

We perform a backtrack search. If there are already n lines in the spread, we choose the $n+1$-st

Figure 1: The Hadamard 2-(63,31,15) design

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one among the lines containing the first point, which is in none of the n spread lines. We try these lines in lexicographic order defined on the numbers of the points they contain. Thus from the fourth spread line on, the spread lines are lexicographically ordered, and any spread we construct is lexicographically greater than the ones constructed before it.

The main problem here is the very big number of isomorphic solutions. Obtaining a spread, we have to check for the existence of an automorphism mapping it into an already constructed one, i.e. into a lexicographically smaller one. $PG(5,2)$, however, has 20158709760 automorphisms, and trying all of them makes the computation time impossible.

The number of automorphisms which can map the constructed spreads into one another, is actually smaller, because the first three lines in them are the same.

Let S_1 and S_2 be two equivalent spreads we have constructed, and let $\alpha \in G$ map the lines of S_1 into the lines of S_2. Let $\alpha\langle i,j\rangle = \langle 1,2\rangle$, $\alpha\langle l,m\rangle = \langle 4,5\rangle$ and $\alpha\langle x,y\rangle = \langle 16,17\rangle$. Suppose there exists $\beta \in G$, and $\beta\langle i,j\rangle = \langle 1,2\rangle$, $\beta\langle l,m\rangle = \langle 4,5\rangle$ and $\beta\langle x,y\rangle = \langle 16,17\rangle$.

There exists $\varphi \in G$, such that $\alpha = \varphi\beta$. Then $\varphi = \alpha\beta^{-1}$. Therefore φ fixes lines $\langle 1,2\rangle$, $\langle 4,5\rangle$, and $\langle 16,17\rangle$.

Before starting the spread construction, for each line of $PG(5,2)$ we find and save an automorphism that maps it into $\langle 1,2\rangle$. For each disjoint with $\langle 1,2\rangle$ line we find and save an automorphism that fixes $\langle 1,2\rangle$ and maps it into $\langle 4,5\rangle$. For each line with no points in P_1 we find and save an automorphism that fixes P_1 and maps the line into $\langle 16,17\rangle$.

When we obtain a new spread, we check if one of the automorphisms $\varphi_n\psi_m\delta_j\gamma_i$ maps it into a lexicographically smaller one. Here γ_i is an automorphism mapping line A_i into $\langle 1,2\rangle$, δ_j is an automorphism fixing $\langle 1,2\rangle$ and mapping line $\gamma_i B_j$ into $\langle 4,5\rangle$, ψ_m is an automorphism fixing P_1 and mapping line $\delta_j\gamma_i C_m$ into $\langle 16,17\rangle$, and φ_n is an automorphism fixing the lines $\langle 1,2\rangle$, $\langle 4,5\rangle$, and $\langle 16,17\rangle$. Here A_i, B_j and C_m are spread lines, $i = 1,2,...,21$, $j = 1,2,...,20$, $m = 1,2,...,s \leq 16$. The automorphism group preserving the three lines $\langle 1,2\rangle$, $\langle 4,5\rangle$, and $\langle 16,17\rangle$ is of order 216, and thus $n = 1,2,...,216$. This way instead of trying all the 20158709760 automorphisms, to check for equivalence we use at most $1451520 = 21.20.16.216$ of them, and the computation becomes possible.

3 Classification

We obtain 131044 inequivalent spreads and classify them with respect to the number of spread lines from the different 3-dimensional subspaces, the automorphism groups which preserve the spreads, and the ranks of the corresponding affine 2-(64,16,5) designs. All the spreads can be found at the second author's web page http://www.moi.math.bas.bg/˜svetlana.

3.1 Spread lines on each 3-dimensional subspace

We establish that up to equivalence there is exactly one regular spread. There are 238 different distributions of the spread lines among the three-dimensional subspaces. They are presented in Table 1. Each one is given in format (s n_2 n_3 n_4 n_5), where s is the number of spreads with this distribution, n_i is the number of three-dimensional subspaces containing i spread lines ($i = 0, 1, 2, 3, 4, 5$). If the last several numbers in this format are all zeros, they are omitted. Thus for the regular spread we write $(1, 0, 0, 0, 21)$, and for the 6 inequivalent spreads with no more than two spread lines in one and the same 3-dimensional subspace $(6, 210)$. For each of these spread types, n_1 and n_0 are not given, because they can be computed by the formulae: $n_1 = 735 - 5n_5 - 4n_4 - 3n_3 - 2n_2$ and $n_0 = 651 - n_5 - n_4 - n_3 - n_2 - n_1$.

3.2 Subgroup of $PGL(6, 2)$ which maps the spread into itself

The orders of the groups of automorphisms, which preserve the spreads, are presented in Table 2, and one spread of each automorphism group order is presented in Table 3, where *aut* is the order of the spread automorphism group, r the rank of the related 2-(64,16,5) design, n_i, $i = 2, 3, 4, 5$ the number of 3-dimensional subspaces with i spread lines, and the lines are presented by two of their points.

3.3 Rank of the R-derived 2-(64,16,5) designs

The 2-rank of the corresponding affine 2-(64,16,5) designs was computed. There are only 2 spreads corresponding to designs of minimal rank 16. The same two spreads were constructed in [11]. All the present results are given in Table 4.

4 Correctness of the results

The construction and classification of the spreads, and the computation of the automorphism groups was done by our own C++ programmes, some of them written, and others modified for this specific task. Since mistakes are always possible, we checked the correctness of the results in the following ways

4.1 Parallel calculations for several disjoint subproblems

Since each 3-dimensional subspace can be mapped by some automorphism into P_1, we can consider the following disjoint cases

Table 1: Classification of the spreads by the number of three-dimensional subspaces containing 2, 3, 4, or 5 spread lines

s,n_2,n_3,n_4,n_5	s,n_2,n_3,n_4,n_5	s n_2,n_3,n_4,n_5	s,n_2,n_3,n_4,n_5	s,n_2,n_3 n_4,n_5	s,n_2,n_3,n_4,n_5
1,64,24,4,5	1,70,24,3,5	1,88,24,0,5	1,60,18,6,6	1,48,0,12,9	2,100,22,4,2
1,100,20,5,2	1,90,18,6,3	4,100,18,6,2	2,112,18,4,2	1,90,6,12,3	1,112,12,7,2
2,112,24,1,2	3,112,26,0,2	2,106,24,2,2	1,100,26,2,2	1,118,24,0,2	1,108,18,3,3
1,136,18,0,2	2,100,24,3,2	1,106,22,3,2	1,124,16,3,2	1,106,18,5,2	2,94,12,10,2
1,94,20,6,2	1,126,0,9,3	1,64,32,0,5	1,64,0,16,5	1,0,0,0,21	1,160,0,0,5
47,134,20,1,1	3,146,12,3,1	33,128,22,1,1	2,152,12,2,1	13,134,16,3,1	3,152,6,5,1
12,122,24,1,1	17,134,18,2,1	7,116,22,3,1	19,122,22,2,1	10,122,20,3,1	2,104,28,2,1
22,140,18,1,1	25,128,20,2,1	5,116,26,1,1	8,140,16,2,1	1,104,18,7,1	2,110,16,7,1
7,104,24,4,1	2,104,26,3,1	11,116,24,2,1	13,122,18,4,1	4,128,18,3,1	4,122,16,5,1
2,116,20,4,1	11,146,16,1,1	5,116,18,5,1	4,128,16,4,1	1,164,8,2,1	4,110,18,6,1
1,146,14,2,1	4,110,24,3,1	1,110,22,4,1	1,104,8,12,1	2,110,26,2,1	1,80,24,8,1
1,110,12,9,1	2,128,12,6,1	1,164,10,1,1	1,140,12,4,1	2,110,28,1,1	1,104,16,8,1
2,176,0,4,1	6,152,14,1,1	2,104,32,0,1	68,140,20,0,1	60,152,16,0,1	31,134,22,0,1
66,146,18,0,1	2,116,28,0,1	28,128,24,0,1	9,158,12,1,1	27,164,12,0,1	12,158,14,0,1
1,122,26,0,1	2,170,8,1,1	5,170,10,0,1	7,176,8,0,1	1,134,12,5,1	1,110,20,5,1
2,98,18,8,1	1,152,8,4,1	1,176,6,1,1	1,80,0,20,1	1,182,6,0,1	2,200,0,0,1
1,132,12,7	14,132,16,5	1,126,12,8	4,126,16,6	41,138,16,4	104,135,21,2
308,141,19,2	124,138,18,3	100,150,14,3	558,147,17,2	55,132,20,3	143,144,16,3
468,144,18,2	57,135,19,3	603,150,16,2	120,141,17,3	5,132,14,6	46,132,18,4
7,129,17,5	8,135,15,5	18,144,14,4	29,135,17,4	2,120,14,8	6,144,12,5
1,138,10,7	1,147,11,5	1,129,9,9	1792,168,12,1	1185,147,19,1	2645,162,14,1
2182,165,13,1	1884,150,18,1	2920,159,15,1	2465,153,17,1	2848,156,16,1	108,165,11,2
733,144,20,1	66,132,22,2	111,147,15,3	489,153,15,2	406,156,14,2	36,153,13,3
269,159,13,2	32,156,12,3	159,162,12,2	12,159,11,3	215,138,20,2	1,141,13,5
18,129,21,3	207,138,22,1	58,132,24,1	24,141,15,4	10,126,22,3	394,141,21,1
16,129,25,1	73,135,23,1	10,162,10,3	3,114,24,4	3,129,15,6	779,174,10,1
11,150,12,4	9,138,14,5	8,129,19,4	6,120,24,3	3,123,21,4	7,147,13,4
7,165,9,3	56,168,10,2	1,114,12,10	5,126,18,5	2,150,10,5	2,138,12,6
1109,171,11,1	427,177,9,1	277,180,8,1	3,171,7,3	4,123,23,3	12,126,24,2
2,168,8,3	17,129,23,2	9,126,26,1	137,183,7,1	63,186,6,1	32,171,9,2
5,120,26,2	23,174,8,2	22,189,5,1	1,153,11,4	1,108,26,4	1,117,27,2
3,156,10,4	1,120,18,6	6,126,20,4	1,156,8,5	4,174,6,3	1,114,26,3
7,177,7,2	1,123,19,5	1,120,22,4	1,114,16,8	8,192,4,1	1,183,5,2
1,123,17,6	2,123,25,2	5,180,6,2	3,198,2,1	4,186,4,2	1,120,20,5
1,180,4,3	1,114,20,6	1,114,22,5	1,120,12,9	1,177,5,3	515,144,22
1016,147,21	187,141,23	3598,153,19	2034,150,20	5427,156,18	7633,159,17
9663,162,16	12194,168,14	26,135,25	11342,165,15	94,138,24	10563,174,12
11707,171,13	2,126,28	8613,177,11	1,129,27	3135,186,8	6834,180,10
4768,183,9	1861,189,7	1040,192,6	492,195,5,	12,132,26	207,198,4
57,201,3,	20,204,2	5,207,1	6,210		

Table 2: Order of the automorphism groups preserving the spreads

order	1	2	3	4	5	6	7	8	9	10	12	15	16	18	21	24	32	36	42
designs	128474	2108	173	84	17	96	3	17	4	1	19	1	3	7	1	7	3	5	1

order	48	64	72	96	108	192	288	324	384	480	576	1152	1728	5760	362880
designs	3	1	1	4	1	1	1	1	1	1	1	1	1	1	1

Table 3: Spreads with different automorphism group orders (a-b stands for $\langle a, b\rangle$)

aut	r	n_2	n_3	n_4	n_5	Lines (all contain lines $\langle 1,2\rangle, \langle 4,5\rangle, \langle 7,8\rangle, \langle 10,11\rangle$ and $\langle 16,17\rangle$)
1	22	134	20	1	1	13-14 19-20 22-23 25-29 26-28 27-41 30-31 32-40 33-36 34-48 35-46 37-45 42-44 49-50 52-53 61-62
2	22	100	20	5	2	13-14 19-20 22-23 25-26 28-29 31-32 34-35 37-40 38-41 39-42 43-44 46-47 49-50 52-56 53-55 54-57
3	22	128	20	2	1	13-14 19-20 22-23 25-29 26-28 27-49 30-31 32-40 33-41 34-44 35-50 36-51 37-45 39-42 46-47 52-58
4	20	112	26	0	2	13-14 19-20 22-23 25-26 28-29 31-32 34-41 35-42 36-47 37-40 38-48 39-46 43-44 49-50 54-56 57-60
5	21	140	20	0	1	13-14 19-20 22-29 23-30 24-31 25-37 26-35 27-34 28-43 32-45 33-41 36-47 38-44 39-46 49-50 52-58
6	21	100	22	4	2	13-14 19-20 22-23 25-26 28-29 31-32 34-35 37-40 38-41 39-42 43-44 46-47 49-50 52-53 55-56 58-59
8	19	128	16	4	1	13-14 19-20 22-23 25-29 26-28 27-44 30-45 31-32 34-35 37-38 40-41 46-47 49-50 52-58 54-56 57-60
9	22	129	9	9	0	13-19 14-20 15-21 22-23 25-26 28-33 29-48 30-51 31-50 34-49 35-46 36-47 38-44 39-43 40-41 53-57
10	22	170	10	0	1	13-14 19-28 20-30 21-29 22-32 23-39 24-49 25-51 26-43 27-38 31-42 33-48 34-35 37-47 41-45 53-60
12	21	112	18	4	2	13-14 19-20 22-23 25-26 28-29 31-32 34-41 35-42 36-40 37-47 38-48 39-46 43-44 49-50 55-56 61-62
16	21	128	24	0	1	13-14 19-20 22-29 23-30 24-31 25-33 26-32 27-28 34-44 35-43 36-51 37-45 38-49 39-50 52-58 53-55
18	22	90	18	6	3	13-14 19-20 22-23 25-26 28-29 31-32 34-35 37-40 38-44 39-50 41-43 42-45 46-47 52-53 55-58 57-60
24	19	104	24	4	1	13-14 19-20 22-23 25-29 26-28 27-41 30-51 31-32 34-35 37-38 40-46 42-44 45-48 52-53 55-56 61-62
32	18	152	16	0	1	13-14 19-20 22-29 23-28 24-41 25-33 26-31 27-48 30-40 32-47 34-35 37-38 43-44 49-50 52-58 53-55
36	20	70	24	3	5	13-14 19-20 22-23 25-26 28-29 31-32 34-35 37-38 40-41 43-44 46-47 49-50 52-53 55-58 56-59 57-60
48	19	104	8	12	1	13-14 19-20 22-23 25-29 26-28 27-49 30-47 31-32 34-35 37-38 40-44 41-45 42-48 52-55 53-60 54-57
64	18	128	16	4	1	13-14 19-20 22-29 23-31 24-41 25-33 26-28 27-48 30-34 32-35 36-40 37-47 43-44 49-50 53-57 55-58
72	19	64	24	4	5	13-14 19-20 22-23 25-26 28-29 31-32 34-35 37-38 40-41 43-44 46-47 49-50 52-53 55-56 58-59 61-62
96	17	152	8	4	1	13-14 19-20 22-29 23-36 24-31 25-33 26-37 27-28 30-45 32-51 34-44 35-50 38-49 39-43 52-56 54-59
108	20	60	18	6	6	13-14 19-20 22-23 25-26 28-29 31-32 34-35 37-38 40-41 43-47 44-46 45-48 52-55 53-57 54-56 58-59
192	18	176	0	4	1	13-14 19-28 20-30 21-34 22-45 23-46 24-33 25-36 26-39 27-41 29-35 31-50 32-47 38-49 42-48 54-56
288	18	88	24	0	5	13-14 19-20 22-23 25-26 28-29 31-32 34-35 37-38 40-41 43-44 46-47 49-50 52-58 53-55 54-56 57-60
324	22	126	0	9	3	13-14 19-20 22-23 25-26 28-29 31-50 32-40 33-44 34-41 35-46 36-45 37-47 38-49 39-43 42-48 53-60
384	17	64	32	0	5	13-14 19-20 22-23 25-26 28-42 29-41 30-40 31-46 32-47 33-48 34-44 35-43 36-45 37-51 38-49 39-50
480	18	200	0	0	1	13-14 19-28 20-30 21-36 22-51 23-39 24-38 25-33 26-43 27-49 29-41 31-34 32-47 35-42 45-48 53-55
576	18	80	0	20	1	13-14 19-28 20-30 21-29 22-45 23-46 24-41 25-37 26-39 27-38 31-43 32-47 33-49 34-35 42-48 54-57
1152	17	64	0	16	5	13-14 19-20 22-23 25-26 28-42 29-41 30-51 31-46 32-45 33-48 34-44 35-43 36-47 37-40 38-49 39-50
1728	18	48	0	12	9	13-14 19-20 22-23 25-26 28-29 31-32 34-35 37-38 40-44 41-45 42-48 43-47 52-55 53-60 54-57 56-59
5760	16	160	0	0	5	13-14 19-20 22-23 25-26 28-43 29-41 30-40 31-50 32-47 33-48 34-44 35-42 36-45 37-51 38-49 39-46
362880	16	0	0	0	21	13-14 19-20 22-23 25-26 28-42 29-48 30-51 31-46 32-45 33-41 34-49 35-43 36-47 37-40 38-44 39-50

aut	r	n_2	n_3	Lines (all contain lines $\langle 1,2\rangle, \langle 4,5\rangle$, and $\langle 16,17\rangle$ and $n_4 = n_5 = 0$)
7	21	147	21	7-8 10-19 11-20 12-21 13-23 14-36 15-22 24-41 25-47 26-46 27-31 28-33 29-44 30-34 32-40 38-48 49-50 53-60
15	22	195	5	7-8 10-19 11-22 12-38 13-26 14-49 15-30 20-50 21-36 23-31 24-33 25-40 27-28 29-32 34-41 35-42 37-45 44-46
21	20	210	0	7-19 8-20 9-22 10-25 11-34 12-24 13-43 14-38 15-39 21-36 23-37 26-50 27-31 28-29 32-40 33-44 35-42 56-59
42	19	126	28	7-8 10-19 11-20 12-37 13-26 14-25 15-22 21-45 23-39 24-41 27-38 29-49 30-40 31-46 32-47 33-44 34-35 42-48

1) P_1 has 2 spread lines, and none of the other 3-dimensional subspaces has more than 2 spread lines

2) P_1 has 3 spread lines, and none of the other 3-dimensional subspaces has more than 3 spread lines

3) P_1 has 4 spread lines, and none of the other 3-dimensional subspaces has more than 4 spread lines

4) P_1 has 5 spread lines

In the first case we work as in the general case, but apply an additional checking of the number of spread lines in each 3-dimensional subspace.

In the second case, we do everything as in the first case, but also notice that for any line of P_1, which

Table 4: Rank of the corresponding 2-(64,16,5) designs

rank	16	17	18	19	20	21	22	all
designs	2	3	14	29	192	6101	124703	131044

is disjoint with $\langle 1,2\rangle$ and $\langle 4,5\rangle$ there exists an automorphism fixing $\langle 1,2\rangle$ and $\langle 4,5\rangle$, and mapping this line into $\langle 7,8\rangle$. We find and save these automorphisms before starting the construction of the spreads.

This allows us to only construct spreads containing the four lines: $\langle 1,2\rangle$, $\langle 4,5\rangle$, $\langle 7,8\rangle$ and $\langle 16,17\rangle$.

When we obtain a new spread, we check if one of the automorphisms $\varphi_n\psi_m\pi\delta_j\gamma_i$ maps it into a lexicographically smaller one. Here γ_i, δ_j, ψ_m and respectively i, j, m, are as in the general case, π is an automorphism fixing $\langle 1,2\rangle$ and $\langle 4,5\rangle$, and mapping the third spread line of P_1 into $\langle 7,8\rangle$. In this case φ_n is an automorphism fixing the lines $\langle 1,2\rangle$, $\langle 4,5\rangle$, $\langle 7,8\rangle$ and $\langle 16,17\rangle$. The automorphism group preserving these four lines is of order 36 ($n = 1, 2, ..., 36$).

In the third case, we work in a similar way. We only construct spreads containing the five lines: $\langle 1,2\rangle$, $\langle 4,5\rangle$, $\langle 7,8\rangle$, $\langle 10,11\rangle$ and $\langle 16,17\rangle$. The automorphism group preserving these five lines is of order 18.

In the fourth case, we construct spreads containing the six lines: $\langle 1,2\rangle$, $\langle 4,5\rangle$, $\langle 7,8\rangle$, $\langle 10,11\rangle$, $\langle 13,14\rangle$ and $\langle 16,17\rangle$. The automorphism group preserving them is the same as in the third case.

In all cases we further reduce the number of the considered automorphisms by applying only γ_i, for which A_i is in at least one 3-dimensional subspace with as many spread lines as P_1. We also apply only δ_j, for which $\langle A_i, B_j\rangle$ has as many spread lines as P_1.

Our C++ programmes for the subcases work much faster (1 day on a 3GHz PC together for the 4 cases) than the one for the general case (10 days on a 3GHz PC). The results coincide and are in Table 5, where for each case we present the number of spreads with ranks r of the corresponding 2-(64,16,5) designs, and the number of different distributions of the spread lines among the three-dimensional subspaces.

Table 5: Distributions and rank results for the four subcases

case	distributions	r=16	r=17	r=18	r=19	r=20	r=21	r=22	all spreads
1	1					1		5	6
2	28				2	50	3843	99151	103046
3	113				2	96	2109	25089	27296
4	96	2	3	14	25	45	149	458	696
total	238	2	3	14	29	192	6101	124703	131044

4.2 Some parallel calculations with different software

- We used software for constructing design resolutions to find the whole number of different solutions for the first parallel class of a resolution of the related STS(63) if the first 6 blocks are fixed to $\{1,2,3\}$, $\{4,5,6\}$, $\{7,8,9\}$, $\{10,11,12\}$, $\{13,14,15\}$ and $\{16,17,18\}$. This number is 15296512 and is the same as the number of all spreads (including isomorphic ones), which we obtain in the fourth case (when we suppose that there is at least one 3-dimensional subspace with 5 spread lines). This is the case with the least number of solutions, because of the greatest number (6) of fixed spread lines.

- The automorphism groups preserving some of the lines are of great importance for our computations, as the rejection of equivalent solutions depends on them. That is why we calculated their orders by two different algorithms independently worked out by both authors.

4.3 Comparison with previous results

The previous results we know, coincide with our results.

Mavron, McDonough and Tonchev construct more than 30000 line spreads of $PG(5,2)$ [11] and establish that there are spreads corresponding to 2-(64,16,5) designs of ranks between 16 and 22. They also find out two different spreads corresponding to 2-(64,16,5) designs of rank 16. Our results show that all spreads yield R-derived 2-(64,16,5) designs of ranks between 16 and 22, and that there are exactly two 2-(64,16,5) designs of rank 16.

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