On a Hypergraph Approach to Multistage Group Testing Problems.

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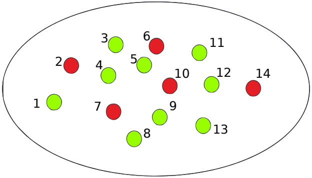
Problem Statement

Let $T = \{1, 2, \ldots, t\}$ be a set of objects and $S_{un} \subset T$, $|S_{un}| \leq s$, be a set of defective elements. Our goal is to find the set S_{un} by performing the minimal number of tests on a chosen subsets of T. The answer to the test $S \subset T$ is positive iff $S \cap S_{un} \neq \varnothing$.

Combinatorial Group Testing



Example:
$t = 14$,
$S_{un} = \{2, 6, 7, 10, 14\}$.

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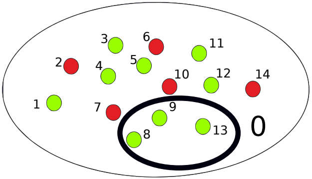
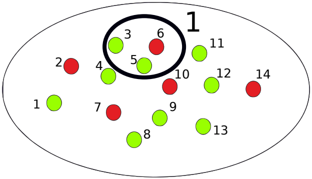


Example:
$t = 14$,
$S_{un} = \{2, 6, 7, 10, 14\}$.
$S = \{8, 9, 13\}$.
$S \cap S_{un} = \varnothing$, thus the test result is negative(0).

Combinatorial Group Testing



Example:
$t = 14$,
$S_{un} = \{2, 6, 7, 10, 14\}$.
$S = \{3, 5, 6\}$.
$S \cap S_{un} = \{6\}$, thus the test result is positive(1).

Different types of search algorithms

- adaptive - later tests depend on the results of previous tests
- nonadaptive - all tests are carried out in parallel.
  Example: disjunctive codes
- multistage - algorithm consists of the several stages, where tests of stage i depend on the results of tests from stages $1, 2, \ldots i-1$.
  Example: list decoding disjunctive codes for 2 stages.

Matrix representation

Any non-adaptive algorithm consisting of N tests can be represented by a binary $N \times t$ matrix X such that each test corresponds to the row, and each element stands for the column.

We put $x_i(j) = 1$ if the j-th element is included in i-th test;
otherwise, $x_i(j) = 0$.

Outcomes of tests can be represented by a binary vector $r(X, S_{un})$.

By $N^p(t, s)$ we denote minimal number of tests in an algorithm, which finds s defects among t elements using p stages.

Hypergraph

Definition

A *hypergraph is a pair* $H = (V, E)$ *such that* $E \subset 2^V \setminus \varnothing$, *where* V *is a set of vertices and* E *is a set of hyperedges.*

Chromatic number

Definition

A *coloring* of H is a map $\varphi : V \to \mathbb{N}$ such that each hyperedge $e \in E$ contains at least two vertices $u, v \in e$ of distinct colors $\varphi(u) \neq \varphi(v)$. The corresponding *chromatic number* $\chi_s(H)$ is the least number of colors for which H has a proper coloring.

Strong Coloring

Definition

A *strong coloring* of H is a map $\varphi : V \to \mathbb{N}$ such that whenever $u, v \in e$ for some $e \in E$, we have that $\varphi(u) \neq \varphi(v)$. The corresponding *strong chromatic number* $\chi_s(H)$ is the least number of colors for which H has a proper strong coloring.

Suppose that we have already performed some set of tests X.

The set of vertices of hypergraph $H(X, S_{un}) = (T, E)$ is equal to the set of elements T. The set of edges E equals to the set of all possible sets of defects, i.e, all subsets $S \subset T$, $|S| \leq s$, such that $r(X, S_{un}) = r(X, S)$.

Previous Results

Bounds from disjunctive codes

$$c_1 \frac{s^2}{\log_2 s} \log_2 t \leq N^1(t, s) \leq c_2 s^2 \log_2 t$$

If X is a matrix of tests of non-adaptive algorithm, then hypergraph $H(X, S_{un})$ has only one hyperedge.

Previous Results

Bounds from list-decoding disjunctive codes

$$s \log_2 t(1 + o(1)) \le N^2(t, s) \le cs \log_2 t(1 + o(1)), t \to \infty$$

If X is a matrix of tests corresponding to the list-decoding disjunctive code, then hypergraph $H(X, S_{un})$ has only constant (independent of t) number of hyperedges.

Previous Results

Specific two stage algorithm for s=2

There exists a $N \times t$ matrix X, $N = 2\log_2 t$, such that the graph $H(X, S_{un})$ has $\sqrt{t}$ vertices with degree 1 and $t - \sqrt{t}$ isolated vertices.

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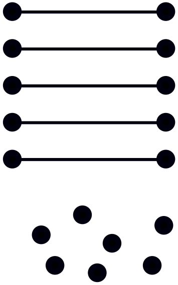

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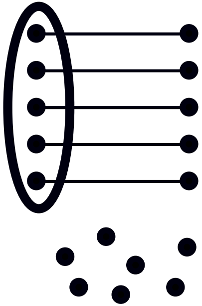

We can find one defective element among $\sqrt{t}/2$ objects using $0.5\log_2 t$ tests.

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Corollary

$$N^2(t, 2) \leq 2.5\log_2 t(1 + o(1)), t \to \infty$$

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For $s = 2$ list decoding disjunctive code give algorithm with the number of tests

$$N^2_{LD}(t,2) \approx 3.11\log_2 t(1 + o(1)).$$

Goal

Information theory bound

$$N^p(t, s) \geq s \log_2 t(1 + o(1)), t \to \infty$$

Adaptive algorithm

$$N^\infty(t, s) = s \log_2 t(1 + o(1)), t \to \infty$$

Goal

Find p such that

$$N^p(t, s) = s \log_2 t(1 + o(1)), t \to \infty$$

4 stage procedure

First stage

Let X be a $N \times t$ matrix of tests of the first stage and $H(X, S_{un}) = (V, E)$ is a corresponding hypergraph. Find the strong chromatic number $\chi_s(H)$ such that there exist disjoint sets $V_1, V_2, \ldots, V_k$, $V = V_1 \bigcup V_2 \bigcup \ldots \bigcup V_k$, $|V_i \bigcap e| \leq 1$ for all $e \in E$.

Note that each set V_i has at most one defective element.

Second stage

Test each set V_i individually.

Here we find the cardinality of the set S_{un} and the set $\{V_{i_1}, V_{i_2}, \ldots, V_{i_{|S_{un}|}}\}$, each of which contains one defective element.

Third stage

Find defective element in the set V_{i_1} by carrying out $\lceil \log_2 |V_{i_1}| \rceil$ tests.

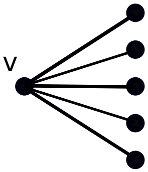
Observe that actually by performing $\sum\limits_{j=1}^{\mathcal{S}_{un}} \lceil \log_2 |V_{i_j}| \rceil$ tests we could identify all defects $\mathcal{S}_{un}$ on this stage.

Fourth stage

Consider all hyperedges $e \in E$, such that e contains the found vertex v and consists of vertices of $v \cup V_{i_2} \cup \ldots \cup V_{i_{|\mathcal{S}_{un}|}}$. For each such e test the set $T \setminus e$.

Fourth stage

Consider all hyperedges $e \in E$, such that e contains the found vertex v and consists of vertices of $v \cup V_{i_2} \cup \ldots \cup V_{i_{|\mathcal{S}_{un}|}}$. For each such e test the set $T \backslash e$.



Case s=2

We can use $\lceil \log_2 \deg(v) \rceil$ tests instead of $\deg(v)$.

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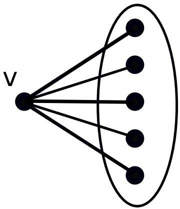

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Total number of tests

Let t' be a number of non-isolated vertices in hypergraph H, and d be a maximal degree of vertex from V. Then the total number of tests can be bounded by

$$N + \chi_s(H) + \lceil \log_2 t' \rceil + d.$$

Total number of tests for s=2

$$N + \chi_s(H) + \lceil \log_2 t' \rceil + \lceil \log_2 d \rceil .$$

Construction for s=2

Let D be the set of all binary words with length N_1 such that the number of ones in each codeword is fixed and equals wN_1. Let C be the q-ary code, $q = \binom{N_1}{wN_1}$, consisting of all q-ary words of length N_1 and having size $t = q^{N_2}$. Let X be a binary $N_1 N_2 \times q^{N_1}$ matrix of a concatenated code with inner code D and outer code C.

Theorem

Chromatic number $\chi(H(X, S_{un}))$ is less or equal to q. For $s = 2$, the product of the maximal degree and the number of non-isolated vertices of H is estimated as follows

$$t' \cdot d \le \max_{w \le \hat{w} \le 2w} \left(\binom{\hat{w} N_1}{w N_1} \cdot \binom{w N_1}{(2w - \hat{w}) N_1} \right)^{N_2}.$$

The optimal choice of parameters gives the algorithm with total number of tests

$$T = N_1 N_2 + q + \lceil \log_2 t' \rceil + \lceil \log_2 d \rceil \sim 2 \log_2 t.$$

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Corollary

$$N^4(t, 2) = 2 \log_2 t (1 + o(1)).$$

Construction for s>2

Theorem

$$N^{2s+1}(t, s) \leq (2s - 1)\log_2 t(1 + o(1)).$$

Number of tests for $t \leq 1000$

t	tests	t	tests	t	tests
8-9	8	29-36	14	126-256	20
10-16	10	37-64	15	257-441	22
17-27	12	65-81	16	442-784	24
28	13	82-125	18	785-1000	25

Number of tests for $t = 10^k$

$t = q^{N_1}$	tests	information bound	tests / $\log_2 t$
10^3	26	19	2.609
10^4	33	26	2.483
10^5	41	33	2.468
10^6	48	39	2.408
10^7	56	46	2.408
10^8	64	53	2.408
10^9	71	59	2.375
10^{10}	79	66	2.378
10^{11}	86	73	2.354
10^{12}	94	79	2.358
10^{13}	102	86	2.362
10^{14}	109	93	2.344
10^{15}	117	99	2.348
10^{16}	124	106	2.333
10^{17}	132	112	2.337
10^{18}	139	119	2.325

Number of tests for t with small ratio tests / $\log_2 t$

$q^{N_1} = t$	tests	information bound	tests / $\log_2 t$
$28^2 = 784$	24	19	2.496
$15^3 = 3375$	29	23	2.474
$21^3 = 9261$	32	26	2.428
$28^3 = 21952$	35	28	2.427
$15^4 = 50625$	37	31	2.368
$21^4 = 194481$	41	35	2.334
$21^5 = 4084101$	51	43	2.322
$15^6 = 11390625$	54	46	2.304
$21^6 = 85766121$	60	52	2.277
$21^9 = 794280046581$	89	79	2.251
$21^{11} \approx 3.5 \cdot 10^{14}$	108	96	2.235

Thank you for your attention!