# Finding one of $D$ defective elements in the additive group testing model

Vladimir Lebedev[1]

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[1] joint work with Christian Deppe

# Group Testing

# Definitions

- $[N] := \{1, 2, \ldots, N\}$ set of elements
- $\mathcal{D} \subset [N]$ a set of defective elements with $D = |\mathcal{D}|$.
- $[i, j] := \{x \in \mathcal{N} : i \leq x \leq j\}$
- $2^{[N]}$ the set of all subsets of $[N]$.

The aim of a searcher is to determine a goal set $\mathcal{G} \subset [N]$.
(for example in classical group testing with $\mathcal{G} = \mathcal{D}$.)

# Definitions

The searcher can choose sets (questions) $S_i \subset [N]$ and asks for the values $t(S_i)$ (answers) of a test function $t : 2^{[N]} \to \mathcal{R}$.

## Definition

Let $t$ be a test function, $s = (S_1, S_2, \ldots, S_n)$ be a sequence of sets $S_i \subset [N]$, and $t(s) := (t(S_1), \ldots, t(S_n))$ . We call $(s, t(s), n)$ a test with test length $n$, if the searcher uniquely determine $\mathcal{G}$.

## Classical group testing

$$t^{(Cla)}(S) = \begin{cases} 0 & , \text{ if } |\mathcal{S} \cap \mathcal{D}| = 0 \\ 1 & , \text{ otherwise.} \end{cases} \quad (1)$$

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## Threshold group testing without gap

$$t^{(Thr)}(S) = \begin{cases} 0 & , \text{if } |\mathcal{S} \cap \mathcal{D}| < u \\ 1 & , \text{if } |\mathcal{S} \cap \mathcal{D}| \geq u. \end{cases} \quad (2)$$

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## Additive group testing

$$t^{(Add)}(S) = |\mathcal{S} \cap \mathcal{D}| \quad (3)$$

# Adaptive vs. Nonadaptive

We distinguish between adaptive and nonadaptive tests.

- We call a test nonadaptive if all questions are specified simultaneously.
- A test is called adaptive if all questions are conducted one by one, and outcomes $(t(S_1), \ldots, t(S_{i-1}))$ of previous questions are known at the time of determining the current question $S_i$.

We consider only adaptive tests.

# Problems

## Problem 1

*Let us call a test successfull if for any $\mathcal{D}$ we have $G = \{i\}$, $i \in \mathcal{D}$. We assume that D and n are given. How big can we choose N to ensure a successful test?*

# Problems

## Problem 1

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## Problem 2

*Let us fix some $j \in [1, d]$ and call a test successfull if for any $\mathcal{D}$ we have $G = \{d_j\}$, where $\mathcal{D} = \{d_1, \ldots, d_D\} \subset [N]$ and $d_1 < d_2, \cdots < d_D$. We assume again that D and n are given. How big can we choose N in this case to ensure a successful test?*

# Threshold Group Testing

Denote by $N_{(Thr)}(n, D, u, m)$ the maximal number of elements in a set $\mathcal{N}$ such that the searcher can find $m$ defective elements in $\mathcal{D} \subset \mathcal{N}$ with the test function $t_{(Thr)}$ and test length $n$.
In 2012 Ahlswede/Deppe/Lebedev proved the following

**Theorem**

*If $D \geq u$ then $N_{(Thr)}(n, D, u, 1) = 2^n + D - 1$.*

# Additive Group Testing

Denote by $N_{(Add)}(n, D)$ the maximal number of elements such that the searcher can find one defective element (construct a successfull test for the Problem 1) with test length $n$.

## Theorem

*We have* $N_{(Add)}(n, D) = 2^n + D - 1.$

Denote by $N_{(Add)}(n, D, j)$ the maximal number of elements such that the searcher can find the $j$th defective element (construct a successfull test for the Problem 2) with test length $n$.

**Theorem**

*We have* $N_{(Add)}(n, D, j) = 2^n + D - 1$ *for* $1 \leq j \leq D$.

# Example

The searcher knows:

- One defective is in $[1, 2^{r-1} + 1]$
- One defective is in $[2^{r-1} + 2, 2^r + 2]$ for any fixed natural $r$.

The searcher needs at least $r$ questions for finding one defective.

# Example

Denote $T_0 = [1, 2^{r-1}+1]$, $T_1 = [2^{r-1}+2, 2^r+2]$ and consider arbitrary question $S$, $S \subseteq [1, 2^r+2]$. We have

$$T_{01} = S \bigcap T_0,\ T_{00} = T_0 \setminus T_{01},\ T_{11} = S \bigcap T_1,\ T_{10} = T_1 \setminus T_{11}.$$

We assume that a genius gives the searcher the information how many defectives are in the sets $T_{01}$, $T_{00}$, $T_{11}$, $T_{10}$. We set

$$A = \begin{cases} T_{00} & , \text{if } |T_{01}| \leq |T_{00}| \\ T_{01} & , \text{if } |T_{00}| < |T_{01}|. \end{cases}$$

and

$$B = \begin{cases} T_{10} & , \text{if } |T_{11}| \leq |T_{10}| \\ T_{11} & , \text{if } |T_{10}| < |T_{11}|. \end{cases}$$

For any question $S$ it is possible to get an answer, such that there is one defective in the set $A$ and there is one defective in the set $B$. Therefore $|A| \geq 2^{r-2}+1$ and $|B| \geq 2^{r-2}+1$. Thus by induction the assumption in the example is correct.