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On resolvable and near-resolvable BIB designs and q-ary equidistant codes [1]

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Abstract. Any resolvable BIB design (v, b, r, k, λ) with $\lambda = 1$ induces an optimal equidistant code C_1 with parameters $(n, N, d) = (r, v, r-1)_{q_1}$ where $q_1 = v/k$ and vice versa. We add to this equivalence two more configurations: an optimal equidistant constant composition $(v, v, v-k+2)_{q_2}$ code C_2 with $q_2 = r+1$ and some additional properties and near-resolvable BIB design with parameters $(v, b', r', k-1, k-2)$.

1 Introduction

Let $Q = \{0, 1, ..., q-1\}$. Any subset $C \subseteq Q^n$ is a code denoted by $(n, N, d)_q$ of length n, cardinality $N = |C|$ and minimum (Hamming) distance d. A code C is called *equidistant* if all the distances between distinct codewords are d (see, for example, [5] and references there).

Definition 1. *A* (v, b, r, k, λ) design *(BIB design* (v, k, λ)*) is an incidence structure* (X, B)*, where* $X = \{1, \ldots, v\}$ *is a set of elements and* B *is a collection of* k*-subsets of elements (called* blocks*) such that every two distinct elements are contained in exactly* $\lambda > 0$ *blocks* $(0 < k \leq v)$*.*

The other two parameters of a BIB (v, k, λ) design are $b = vr/k$ (the number of blocks) and $r = \lambda(v-1)/(k-1)$ (the number of blocks containing one element).

In terms of binary incident matrix a (v, k, λ) design is a binary $(v \times b)$ matrix A with columns of weight k such that any two distinct rows contain exactly λ common nonzero positions.

Definition 2. *A* (v, k, λ)*-design* (X, B) *is* resolvable *(called RBIB design) if the set* B *can be partitioned into not-intersecting subsets* B_i*,* $i = 1, \ldots, r$*,*

$$B = \bigcup_{i=1}^{r} B_i,$$

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such that for every i, the set (X, B_i) is a trivial 1-design (i.e. any element of X occurs in B_i exactly one time).

The incident matrix A of a resolvable design (v, k, λ) looks as follows:

$$A = [A_1 \mid \cdots \mid A_r], \tag{1}$$

where for any $i \in \{1, \ldots, r\}$ the every row of A_i has the weight 1.

Definition 3. *A $(v, k, k-1)$-design (X, B) is near-resolvable (NRBIB) if the set B can be partitioned into not-intersecting subsets B_i, $i = 1, \ldots, v$,*

$$B = \bigcup_{i=1}^{v} B_i,$$

such that for every i, the set $(X \setminus \{i\}, B_i)$ is a trivial 1-design (i.e. any element of X (except i) occurs in B_i exactly one time).

The incident matrix A of a near-resolvable design (v, k, λ) can be presented as follows:

$$A = [A_1 \mid \cdots \mid A_v], \tag{2}$$

where for any $i \in \{1, \ldots, r\}$ the every row of the submatrix A_i has the weight 1 with one exception; the ith row of A_i is the zero row.

See [1, 4] and references there for resolvable and near-resolvabe designs.

2 Main results

The following result is known [6].

Theorem 1. *An optimal equidistant $(n, d, N)_q$ code exists if and only if there exists a resolvable (v, k, λ) design, where*

$$q = v/k, \;\; n = \lambda(v-1)/(k-1), \;\; N = v, \;\; d = n - \lambda. \tag{3}$$

For a given q-ary code C with parameters $(n, N, d)_q$ denote by $M = M_C$ the matrix over Q of size $N \times n$ formed by the all codewords of C.

For the case $\lambda = 1$ we can add to Theorem 1 the following

Theorem 2. *The following configurations are equivalent:*

- *(i) A resolvable $(v, k, 1)$ design.*

- *(ii) An optimal equidistant* $(n_1, d_1, N_1)_{q_1}$ *code* C_1 *with parameters*

$$q_1 = v/k, \;\; n_1 = (v-1)/(k-1), \;\; N_1 = v, \;\; d_1 = (v-k)/(k-1).$$

- *(iii) An optimal equidistant constant composition* $(n_2, N_2, d_2)_{q_2}$*-code* C_2 *with parameters*

$$q_2 = (v+k-2)/(k-1), \;\; n_2 = v, \;\; N_2 = v, \;\; d_2 = v-k+2$$

  where every nonzero symbol occurs in every row (respectively, in every column) of the matrix M_2 *exactly* $(k-1)$ *times and with the following property: every two rows of* M *coincide in* $k-2$ *positions, which have the same symbol of the alphabet.*

- *(iv) A near-resolvable* $(v, b', r', k-1, k-2)$ *design, where*

$$b' = v(v-1)/(k-1), \;\; r' = v-1.$$

Denote by $N_q(n,d,w)$ the maximal possible number N of codewords in the $(n,N,d)_q$ code, and by $N_q(n,d,w)$ the maximal possible number N of codewords of weight w in the $(n,N,d)_q$ code.

The equidistant $(n,N,d)_q$ code C is optimal if its cardinality meets the Plotkin bound

$$N_q(n,d) \le \frac{qd}{qd-(q-1)n}, \;\; \text{if } qd > (q-1)n, \tag{4}$$

The code C_1 from Theorem 2 is optimal according to the bound (4).

The equidistant constant weight $(n,N,d)_q$ code C with weight of codewords w is optimal if its cardinality meets the following bound [2]

$$N_q(n,d,w) \le \frac{(q-1)dn}{qw^2-(q-1)(2w-d)n}, \;\; \text{if } qw^2 > (q-1)(2w-d)n, \tag{5}$$

The code C_2 from Theorem 2 is optimal according to the bound (5).

We shortly explain the constructions.
(i) $\leftrightarrow$ (ii) Let $X = \{x_1, x_2, \ldots, x_v\}$ and let $Q = \{0, 1, \ldots, q-1\}$. Given a symbol $i \in Q$ denote by $T(i)$ a binary vector of length q and weight 1 with

$(i+1)$th nonzero position. For a vector $c = (c_1, \ldots, c_n)$ of length n over Q denote by $T(c)$ the binary vector $T(c) = (T(c_1), \ldots, T(c_n))$ of length $q \cdot n$. For a given $(n, N, d)_q$ code C with matrix M, denote by $T(M)$ a binary $(N \times qn)$-matrix obtained from M by applying the operator $T(C)$ to all codewords. It is easy to see that if C is an equidistant $(n, N, d = n-1)_q$ code then the matrix $T(M)$ is an incident matrix A in the form (1) of the resolvable BIB design with parameters $v, b, r, k, \lambda = 1$, satisfying (3). Conversely, given an incident matrix A in the form (1) of the resolvable BIB design with parameters $v, b, r, k, \lambda = 1$, the matrix $T^{-1}(A)$ is the matrix M_1 formed by the all codewords of equidistant code C_1 with parameters n_1, N_1, d_1, q_1 satisfying (3)

(iii) $\leftrightarrow$ (iv) Given a nonzero symbol $i \in Q$ denote by $\Gamma(i)$ a binary vector of length $q-1$ and weight 1 with ith nonzero position. For a vector $c = (c_1, \ldots, c_n)$ of length n over Q denote $\Gamma(c)$ the binary vector $\Gamma(c) = (\Gamma(c_1), \ldots, \Gamma(c_n))$ of length $(q-1) \cdot n$. For a given $(n, N, d)_q$ code C with matrix M denote by $\Gamma(M)$ a binary $(N \times (q-1)n)$-matrix obtained from M by applying the operator $\Gamma(C)$ to all elements. It is easy to see that if C_2 is an equidistant $(n, n, d = n-k+2)_q$ code with properties stated in Theorem 2, then the matrix $\Gamma(M_2)$ is an incident matrix A in the form (2) of the near-resolvable BIB design with parameters $v, b', r', k-1, k-2)$, satisfying (3). Conversely, given an incident matrix A in the form (2) of the near-resolvable BIB design with parameters $v, b, r, k-1, k-2$, the matrix $\Gamma^{-1}(A)$ is the matrix M_2 formed by the all codewords of equidistant code C_2 with parameters and properties stated in Theorem 2.

(i) $\leftrightarrow$ (iii) Given a resolvable BIB design (X, B) with parameters $(v, b, r, k, 1)$, where $X = \{1, 2, \ldots, v\}$, $B = \{z_1, z_2, \ldots, z_b\}$, and

$$B = B_1 \cup B_2 \cup \cdots \cup B_r,$$

we build the q-ary $(v \times v)$-matrix $M = [m_{f,g}]$ over $Q = \{0, 1, \ldots, q_2 - 1\}$ where $q_2 = r + 1$ as follows: to any block $z_\ell = \{i, j, u, \ldots, h\} \in B_s$, we associate the element

$$m_{f,g} = s, \quad \text{for all} \quad f, g \in z_\ell, \quad f \neq g,$$

and $m_{f,f} = 0$ for all $f \in \{1, 2, \ldots v\}$. Then it is easy to see that M is formed by the q-ary equidistant $(v, v, v-k+2)_{q_2}$-code C_2 with properties stated in Theorem 2. Conversely, given an equidistant $(n, n, n-k+2)_q$-code C_2 satisfying Theorem 2 with matrix M, for every jth row $c(j)$ of M, $j \in \{1, \ldots, v\}$, we form $q-1$ blocks $z_{j,1}, \ldots, z_{j,q-1}$ as follows: if $c(j)$ contains $k-1$ elements s in positions $i_1, i_2, \ldots, i_{k-1}$ we form the block $z_{j,s} = \{j, i_1, i_2, \ldots, i_{k-1}\}$ and place this block to the set B_s. In this way we obtain $b = n(q-1)/(k-1)$ blocks of size k partitioned into $r = q-1$ subsets B_s, containing $v = n$ elements $\{1, 2, \ldots, n\}$. It is easy to see that every pair of elements $\{1, 2, \ldots, v\}$ occurs exactly once.

We give an example. Let A_1 be the incident matrix of the resolvable $(16, 4, 1)$ design (or affine plane of order 4) (for shortness, we put only ones and omit zeros):

$$A_1 = \left[\begin{array}{cccc|cccc|cccc|cccc|cccc}
1 & & & & 1 & & & & 1 & & & & 1 & & & & 1 & & & \\
1 & & & & & 1 & & & & 1 & & & & 1 & & & & 1 & & \\
1 & & & & & & 1 & & & & 1 & & & & 1 & & & & 1 & \\
1 & & & & & & & 1 & & & & 1 & & & & 1 & & & & 1 \\
\hline
 & 1 & & & 1 & & & & & 1 & & & & & 1 & & & & & 1 \\
 & 1 & & & & 1 & & & 1 & & & & & & & 1 & & & 1 & \\
 & 1 & & & & & 1 & & & & & 1 & 1 & & & & & 1 & & \\
 & 1 & & & & & & 1 & & & 1 & & & 1 & & & 1 & & & \\
\hline
 & & 1 & & 1 & & & & & & 1 & & & & & 1 & & 1 & & \\
 & & 1 & & & 1 & & & & & & 1 & & & 1 & & 1 & & & \\
 & & 1 & & & & 1 & & 1 & & & & & 1 & & & & & & 1 \\
 & & 1 & & & & & 1 & & 1 & & & 1 & & & & & & 1 & \\
\hline
 & & & 1 & 1 & & & & & & & 1 & & 1 & & & & & 1 & \\
 & & & 1 & & 1 & & & & & 1 & & 1 & & & & & & & 1 \\
 & & & 1 & & & 1 & & & 1 & & & & & & 1 & 1 & & & \\
 & & & 1 & & & & 1 & 1 & & & & & & 1 & & & 1 & &
\end{array}\right].$$

From A_1 using our operator T^{-1} we obtain the optimal equidistant $(5,16,4)_4$ code C_1 (which is common known) and using our construction, we obtain the optimal equidistant constant composition $(16,16,14)_6$ code C_2, whose matrices M_1 and M_2 of codewords we give.

$$M_1 = \left[\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 \\
0 & 2 & 2 & 2 & 2 \\
0 & 3 & 3 & 3 & 3 \\
\hline
1 & 0 & 1 & 2 & 3 \\
1 & 1 & 0 & 3 & 2 \\
1 & 2 & 3 & 0 & 1 \\
1 & 3 & 2 & 1 & 0 \\
\hline
2 & 0 & 2 & 3 & 1 \\
2 & 1 & 3 & 2 & 0 \\
2 & 2 & 0 & 1 & 3 \\
2 & 3 & 1 & 0 & 2 \\
\hline
3 & 0 & 3 & 1 & 2 \\
3 & 1 & 2 & 0 & 3 \\
3 & 2 & 1 & 3 & 0 \\
3 & 3 & 0 & 2 & 1
\end{array}\right], \quad M_2 = \left[\begin{array}{cccc|cccc|cccc|cccc}
0 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 2 & 5 & 3 & 4 & 2 & 4 & 5 & 3 \\
1 & 0 & 1 & 1 & 3 & 2 & 5 & 4 & 5 & 2 & 4 & 3 & 4 & 2 & 3 & 5 \\
1 & 1 & 0 & 1 & 4 & 5 & 2 & 3 & 3 & 4 & 2 & 5 & 5 & 3 & 2 & 4 \\
1 & 1 & 1 & 0 & 5 & 4 & 3 & 2 & 4 & 3 & 5 & 2 & 3 & 5 & 4 & 2 \\
\hline
2 & 3 & 4 & 5 & 0 & 1 & 1 & 1 & 2 & 4 & 5 & 3 & 2 & 5 & 3 & 4 \\
3 & 2 & 5 & 4 & 1 & 0 & 1 & 1 & 4 & 2 & 3 & 5 & 5 & 2 & 4 & 3 \\
4 & 5 & 2 & 3 & 1 & 1 & 0 & 1 & 5 & 3 & 2 & 4 & 3 & 4 & 2 & 5 \\
5 & 4 & 3 & 2 & 1 & 1 & 1 & 0 & 3 & 5 & 4 & 2 & 4 & 3 & 5 & 2 \\
\hline
2 & 5 & 3 & 4 & 2 & 4 & 5 & 3 & 0 & 1 & 1 & 1 & 2 & 3 & 4 & 5 \\
5 & 2 & 4 & 3 & 4 & 2 & 3 & 5 & 1 & 0 & 1 & 1 & 3 & 2 & 5 & 4 \\
3 & 4 & 2 & 5 & 5 & 3 & 2 & 4 & 1 & 1 & 0 & 1 & 4 & 5 & 2 & 3 \\
4 & 3 & 5 & 2 & 3 & 5 & 4 & 2 & 1 & 1 & 1 & 0 & 5 & 4 & 3 & 2 \\
\hline
2 & 4 & 5 & 3 & 2 & 5 & 3 & 4 & 2 & 3 & 4 & 5 & 0 & 1 & 1 & 1 \\
4 & 2 & 3 & 5 & 5 & 2 & 4 & 3 & 3 & 2 & 5 & 4 & 1 & 0 & 1 & 1 \\
5 & 3 & 2 & 4 & 3 & 4 & 2 & 5 & 4 & 5 & 2 & 3 & 1 & 1 & 0 & 1 \\
3 & 5 & 4 & 2 & 4 & 3 & 5 & 2 & 5 & 4 & 3 & 2 & 1 & 1 & 1 & 0
\end{array}\right]$$

Now applying the operator Γ to the matrix M_2 we obtain the binary (16×80) matrix which is the incident matrix A_2 of near-resolvable $(16,3,2)$ design. We give the first 40 columns of this matrix.

$$A_2 = \left[\begin{array}{cccc|cccc|c}
00000 & 10000 & 10000 & 10000 & 01000 & 00100 & 00010 & 00001 & \cdots \\
10000 & 00000 & 10000 & 10000 & 00100 & 01000 & 00001 & 00010 & \cdots \\
10000 & 10000 & 00000 & 10000 & 00010 & 00001 & 01000 & 00100 & \cdots \\
10000 & 10000 & 10000 & 00000 & 00001 & 00010 & 00100 & 01000 & \cdots \\
\hline
01000 & 00100 & 00010 & 00001 & 00000 & 10000 & 10000 & 10000 & \cdots \\
00100 & 01000 & 00001 & 00010 & 10000 & 00000 & 10000 & 10000 & \cdots \\
00010 & 00001 & 01000 & 00100 & 10000 & 10000 & 00000 & 10000 & \cdots \\
00001 & 00010 & 00100 & 01000 & 10000 & 10000 & 10000 & 00000 & \cdots \\
\hline
01000 & 00001 & 00100 & 00010 & 01000 & 00010 & 00001 & 00100 & \cdots \\
00001 & 01000 & 00010 & 00100 & 00010 & 01000 & 00100 & 00001 & \cdots \\
00100 & 00010 & 01000 & 00001 & 00001 & 00100 & 01000 & 00010 & \cdots \\
00010 & 00100 & 00001 & 01000 & 00100 & 00001 & 00010 & 01000 & \cdots \\
\hline
01000 & 00010 & 00001 & 00100 & 01000 & 00001 & 00100 & 00010 & \cdots \\
00010 & 01000 & 00100 & 00001 & 00001 & 01000 & 00010 & 00100 & \cdots \\
00001 & 00100 & 01000 & 00010 & 00100 & 00010 & 01000 & 00001 & \cdots \\
00100 & 00001 & 00010 & 01000 & 00010 & 00100 & 00001 & 01000 & \cdots
\end{array}\right]$$

3 References

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