Fifteenth International Workshop on Algebraic and Combinatorial Coding Theory
June 18-24, 2016, Albena, Bulgaria *pp. 19–26*

Upper bounds on the smallest size of a complete cap in $\mathrm{PG}(3,q)$ and $\mathrm{PG}(4,q)$ [1]

Daniele Bartoli daniele.bartoli@unipg.it
Department of Mathematics Ghent University, Ghent, 9000 Belgium
Dipartimento di Matematica e Informatica, Università degli Studi di Perugia
Via Vanvitelli 1, Perugia 06123 Italy
Alexander A. Davydov, Alexey A. Kreshchuk
{adav, krsch}@iitp.ru
Kharkevich Institute for Information Transmission Problems, Russian Academy of Sciences Bol'shoi Karetnyi per. 19, GSP-4, Moscow, 127994, Russian Federation
Stefano Marcugini, Fernanda Pambianco
{stefano.marcugini,fernanda.pambianco}@unipg.it
Dipartimento di Matematica e Informatica, Università degli Studi di Perugia
Via Vanvitelli 1, Perugia, 06123, Italy

Abstract. In this work we summarize some recent results to be included in a forthcoming paper [2]. We present and analyze computational results concerning small complete caps in the projective spaces $\mathrm{PG}(N,q)$ of dimension $N=3$ and $N=4$ over the finite field of order q. The results have been obtained using randomized greedy algorithms and the algorithm with fixed order of points (FOP). The new complete caps are the smallest known. Basing on them, we obtained new upper bounds on the minimum size $t_2(N,q)$ of a complete cap in $\mathrm{PG}(N,q)$. Our investigations and results allow to conjecture that these bounds hold for all $q \geq 23$.

1 Introduction. The main results

Let $\mathrm{PG}(N,q)$ be the N-dimensional projective space over the Galois field $\mathbb{F}_q$ of order q. A cap $\mathcal{K}$ in $\mathrm{PG}(N,q)$ is a set of points no three of which are collinear. A cap $\mathcal{K}$ is complete if it is not contained in a larger cap. Caps in $\mathrm{PG}(2,q)$ are also called arcs and they have been widely studied, see e.g. [6,8]. If $N>2$ only few constructions and bounds are known, see e.g. [1,3,5–7,9].

[1]The research of D. Bartoli was supported by the European Community under a Marie-Curie Intra-European Fellowship (FACE project: number 626511). The research of A.A. Davydov and A.A. Kreshchuk was carried out at the IITP RAS at the expense of the Russian Foundation for Sciences (project 14-50-00150). The research of D. Bartoli, S. Marcugini and F. Pambianco was supported in part by Ministry for Education, University and Research of Italy (MIUR) (Project "Geometrie di Galois e strutture di incidenza") and by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INDAM).

Caps are connected with Coding Theory. Let $[n,k,d]_q$ be a linear q-ary code with length n, dimension k, and minimum distance d. A linear q-ary code having a parity-check matrix obtained by taking as columns the homogeneous coordinates of the points of a cap in $\mathrm{PG}(N,q)$ has $d=4$ (exceptions are given by the 5-cap in $\mathrm{PG}(3,2)$ and the 11-cap in $\mathrm{PG}(4,3)$. Complete n-caps in $\mathrm{PG}(N,q)$ correspond to non-extandable $[n,n-N-1,4]_q$ codes. For $N=2$ these codes are MDS; if $N=3$ they are Almost MDS.

A central problem concerning caps is to determine the spectrum of the possible sizes of complete caps in a given space. Of particular interest for applications to Coding Theory is the lower part of the spectrum; in fact, small complete caps in projective Galois spaces correspond to quasi-perfect linear codes with minimum distance 4, covering radius 2 and small covering density [6].

We denote by $t_2(N,q)$ the minimum size of a complete cap in $\mathrm{PG}(N,q)$. The exact values of $t_2(N,q)$ are known only for very small q.

The trivial lower bound for $t_2(N,q)$ is $\sqrt{2}q^{(N-1)/2}$. General constructions of complete caps whose size is close to this lower bound are only known for q even [6,7]. Using a modification of the approach of [8], the probabilistic upper bound $t_2(N,q) < cq^{\frac{N-1}{2}}\log^{300} q$, with c constan has been obtained in [4,5]. Computer assisted results on small complete caps in $\mathrm{PG}(N,q)$ and $\mathrm{AG}(N,q)$ are given in [3,6,9]. Here $\mathrm{AG}(N,q)$ is the N-dimensional affine space over $\mathbb{F}_q$.

In this paper we obtain by computer searches results concerning upper bounds on the functions $t_2(3,q)$ and $t_2(4,q)$. These searches requested a huge amount of memory and execution time. We constructed small complete caps in $\mathrm{PG}(3,q)$ and $\mathrm{PG}(4,q)$ using two different approaches[2]: the algorithm with fixed order of points (FOP), for $q \in L_N$ in $\mathrm{PG}(N,q)$, and randomized greedy algorithms, for $q \in G_N$ in $\mathrm{PG}(N,q)$, where $N=3,4$ and

$$\begin{aligned} L_3 &:= \{q \le 4673, q \text{ prime}\} \cup \{5003, 6007, 7001, 8009\},\\ G_3 &:= \{q \le 3701,\ q \text{ prime}\} \cup \{3803, 3907, 4001, 4289\},\\ L_4 &:= \{q \le 1201, q \text{ prime}\} \cup \{1259\},\ G_4 := \{q \le 463,\ q \text{ prime}\}. \end{aligned}$$

Theorem 1. *Let $t_2(N,q)$ be the minimum size of a complete cap in the projective space $\mathrm{PG}(N,q)$. The following upper bounds on $t_2(N,q)$ hold.*

A. *Upper bounds with constant parameters:*

$$t_2(N,q) < q^{\frac{N-1}{2}} \ln^{\frac{N+1}{4}} q, \qquad 23 \le q \in L_N, \qquad N=3,4; \tag{1}$$

$$t_2(N,q) < \frac{N+1}{4} q^{\frac{N-1}{2}} \ln q, \qquad 23 \le q \in L_N, \qquad N=3,4; \tag{2}$$

$$t_2(N,q) < \sqrt{N+2}\cdot q^{\frac{N-1}{2}} \sqrt{\ln q}, \qquad 3 \le q \in L_N, \qquad N=3,4. \tag{3}$$

[2]Calculations were performed using computational resources of Multipurpose Computing Complex of National Research Centre “Kurchatov Institute”, http://computing.kiae.ru

B. *Upper bounds with decreasing parameters:*

$$t_2(N,q) < q^{\frac{N-1}{2}} \ln^{f_N^{up}(q)} q, \qquad 5 \le q \in L_N, \qquad N = 3,4,$$

$$f_3^{up}(q) = 0.7 + \frac{1.15}{\ln(0.3q)}, \qquad f_4^{up}(q) = 0.75 + \frac{1.3}{\ln(0.4q)};$$

$$t_2(N,q) < d_N^{up}(q) \cdot q^{\frac{N-1}{2}} \ln q, \qquad 41 \le q \in L_N, \qquad N = 3,4,$$

$$d_3^{up}(q) = 0.5 + \frac{1.15}{\ln(0.025q)}, \qquad d_4^{up}(q) = 0.7 + \frac{1.05}{\ln(0.08q)};$$

$$t_2(N,q) < \beta_N^{up}(q) \cdot q^{\frac{N-1}{2}} \sqrt{\ln q}, \qquad 3 \le q \in L_N, \qquad N = 3,4,$$

$$\beta_3^{up}(q) = \sqrt{3+1} + \frac{1.1}{\ln(2q)}, \qquad \beta_4^{up}(q) = \sqrt{4+1} + \frac{1.1}{\ln q}.$$

Conjecture 1. In $\mathrm{PG}(N,q)$, $N = 3,4$, the bounds (1)-(3) hold for all $q \ge 23$.

Complete caps obtained in this work are the smallest known in literature for $\mathrm{PG}(3,q)$ with prime $61 \le q \in L_3$ and $\mathrm{PG}(4,q)$ with prime $17 \le q \in L_4$.

2 Algorithms for small caps in $\mathrm{PG}(N,q)$. Graphics

Algorithm with fixed order of points (FOP). This algorithm is a particular type of random algorithm. Firstly, see [1], we fix a particular order on the points of $\mathrm{PG}(N,q)$. The algorithm builds a complete cap step by step adding a new point at each step, until a complete cap is obtained. Let $K^{(i-1)}$ be the cap obtained at the $(i-1)$-th step. Among the points not lying on bisecants of $K^{(i-1)}$, the first point in the fixed order is added to $K^{(i-1)}$ to obtain $K^{(i)}$.

FOP with lexicographical order of points. For simplicity, we considered only q prime. Suppose that the points of $\mathrm{PG}(N,q)$ are ordered as $A_1, A_2, \ldots, A_{\frac{q^{N+1}-1}{q-1}}$. Let the elements of the field $\mathbb{F}_q = \{0,1,\ldots,q-1\}$ be treated as integers modulo q. Let the points A_i of $\mathrm{PG}(N,q)$ be represented in homogenous coordinates so that $A_i = (x_0^{(i)}, x_1^{(i)}, \ldots, x_N^{(i)})$, $x_j^{(i)} \in \mathbb{F}_q$, where the leftmost non-zero element is 1. The points of $\mathrm{PG}(N,q)$ are sorted according to the lexicographic order on the $(N+1)$-tuples of their coordinates. This order is called a *lexicographical order of points*. We call *lexicap* a cap obtained by the algorithm FOP with the lexicographical order of points.

We denote by $t_2^L(N,q)$ the size of a complete lexicap in $\mathrm{PG}(N,q)$.

It is important that for such a lexicographical order for prime q, *the size $t_2^L(N,q)$ of a complete lexicap and its set of points depend on N and q only.*

Randomized greedy algorithms. It is a step by step algorithm. At every step a randomized greedy algorithm maximizes an objective function f and only some steps are executed in a random manner. The number of these steps, their ordinal numbers, and some other parameters of the algorithm have been taken intuitively. Also, if the same maximum of f can be obtained in distinct ways, one way is chosen randomly.

Let $t_2^G(N,q)$ denote the smallest size of a complete cap in $\mathrm{PG}(N,q)$ obtained using greedy algorithms.

A graphical representation of the bounds of Theorem 1 is shown in Figs. 1–4. In Fig. 2, values $f_N(q)$, $f_N^L(q)$, and $f_N^G(q)$ are defined by the equalities $t_2(N,q) = q^{\frac{N-1}{2}} \ln^{f_N(q)} q$, $t_2^L(N,q) = q^{\frac{N-1}{2}} \ln^{f_N^L(q)} q$, and $t_2^G(N,q) = q^{\frac{N-1}{2}} \ln^{f_N^G(q)} q$.

Sizes $t_2^L(N,q)$ and $t_2^G(N,q)$ of complete caps used in this work and other details of the way for the formulation of the bounds are given in [2].

References

[1] D. Bartoli, A.A. Davydov, G. Faina, S. Marcugini, F. Pambianco, New upper bounds on the smallest size of a complete cap in the space $\mathrm{PG}(3,q)$. In *Proc. VII Int. Workshop on Optimal Codes and Related Topics, OC2013, Albena, Bulgaria,* pp. 26–32, 2013. http://www.moi.math.bas.bg/oc2013/a4.pdf

[2] D. Bartoli, A.A. Davydov, A.A. Kreshchuk, S. Marcugini, F. Pambianco, Small complete caps in $\mathrm{PG}(3,q)$ and $\mathrm{PG}(4,q)$, preprint.

[3] D. Bartoli, G. Faina, M. Giulietti, Small complete caps in three-dimensional Galois spaces. *Finite Fields Appl.* **24**, 2013, 184–191.

[4] D. Bartoli, S. Marcugini, F. Pambianco, A probabilistic construction of low density quasi-perfect linear codes, In *Proc. XIV Int. Workshop on Algebraic and Combinatorial Coding Theory, ACCT2014, Svetlogorsk, Russia,* pp. 51–56, 2014. http://www.moi.math.bas.bg/acct2014/a8.pdf

[5] D. Bartoli, S. Marcugini, F. Pambianco, A construction of small complete caps in projective spaces, submitted.

[6] A.A. Davydov, G. Faina, S. Marcugini, F. Pambianco, On sizes of complete caps in projective spaces $\mathrm{PG}(n,q)$ and arcs in planes $\mathrm{PG}(2,q)$. *J. Geom.* **94**, 2009, 31–58.

[7] A.A. Davydov, M. Giulietti, S. Marcugini, F. Pambianco, New inductive constructions of complete caps in $\mathrm{PG}(n,q)$, q even. *J. Combin. Des.* **18**, 2010, 177–201.

[8] J.H. Kim, V. Vu, Small complete arcs in projective planes. *Combinatorica* **23**, 2003, 311–363.

[9] I. Platoni, Complete caps in $\mathrm{AG}(3,q)$ from elliptic curves. *J. Alg. Appl.* **13**, 1450050 (8 pages) 2014.

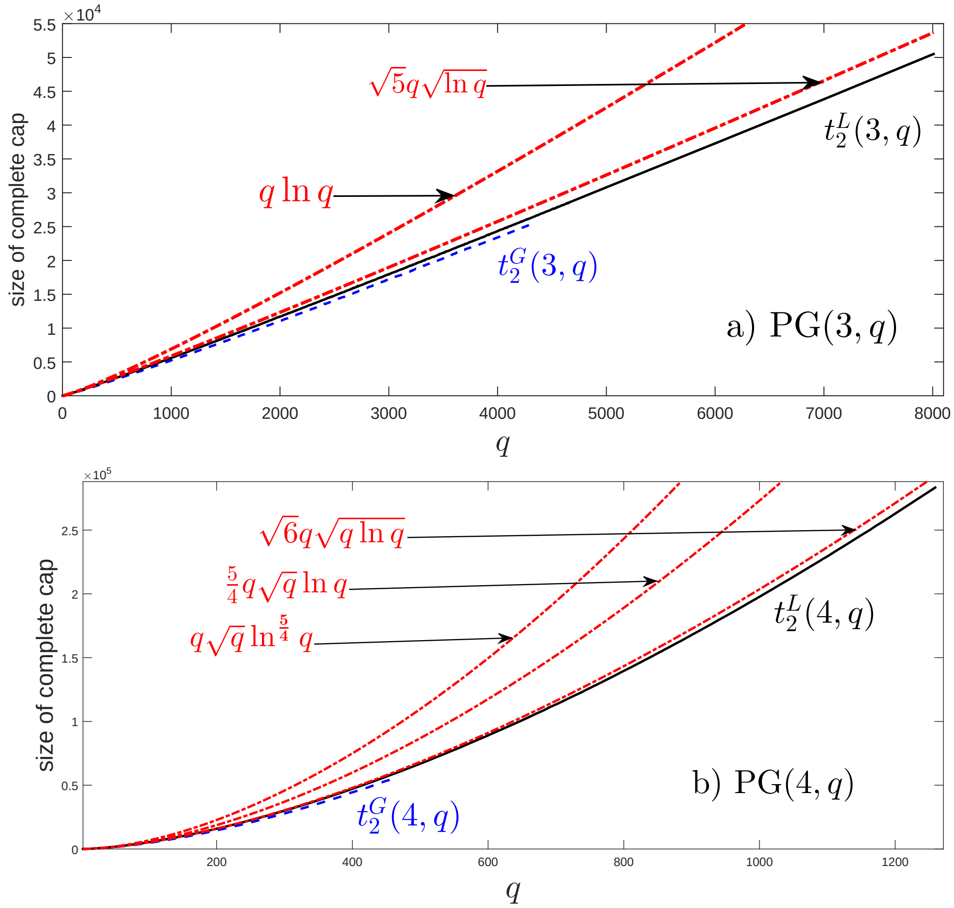
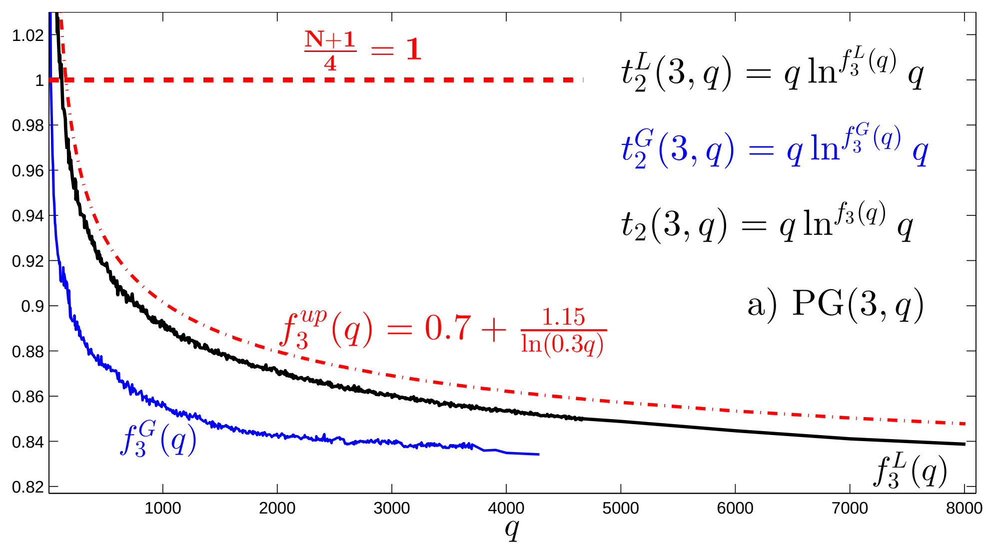


Figure 1: **Upper bounds** $t_2(N,q) < q^{\frac{N-1}{2}} \ln^{\frac{N+1}{4}} q$, $t_2(N,q) < \frac{N+1}{4} q^{\frac{N-1}{2}} \ln q$, and $t_2(N,q) < \sqrt{N+2} \cdot q^{\frac{N-1}{2}} \sqrt{\ln q}$ (*two or three top dashed-dotted curves*) **vs sizes** $t_2^L(N,q)$ of complete lexicaps, $q \in L_N$ (*solid curve*) and **sizes** $t_2^G(N,q)$ of complete caps obtained by greedy algorithms, $q \in G_N$ (*bottom dashed curve*). a) $N = 3$, PG$(3,q)$; b) $N = 4$, PG$(4,q)$



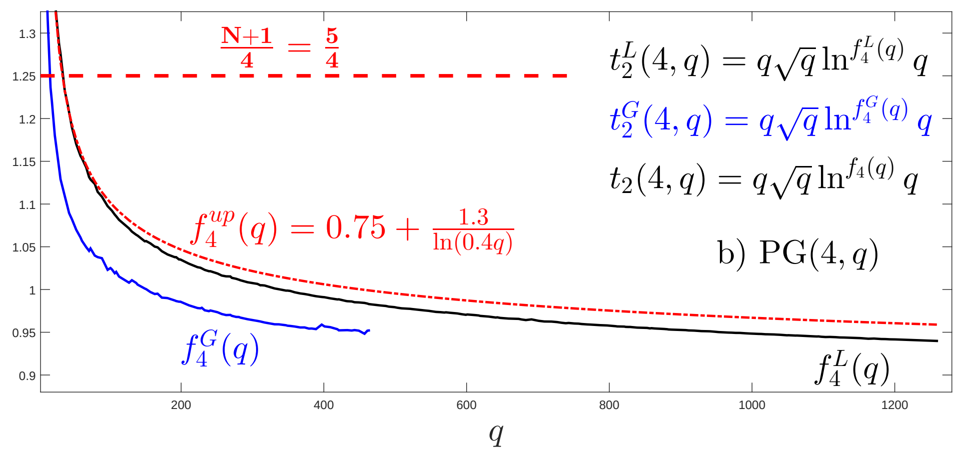


Figure 2: **Upper bounds** $f_N(q) < \frac{N+1}{4}$ (*dashed line* $y = \frac{N+1}{4}$) and $f_N(q) < f_N^{up}(q)$ (*top dashed-dotted curve*) **vs values** $f_N^L(q)$, $q \in L_N$ (*the 2-nd solid curve*) and $f_N^G(q)$, $q \in G_N$ (*bottom solid curve*).
a) $N = 3$, $f_3^{up} = 0.7 + 1.15/\ln(0.3q)$; b) $N = 4$, $f_4^{up} = 0.75 + 1.3/\ln(0.4q)$

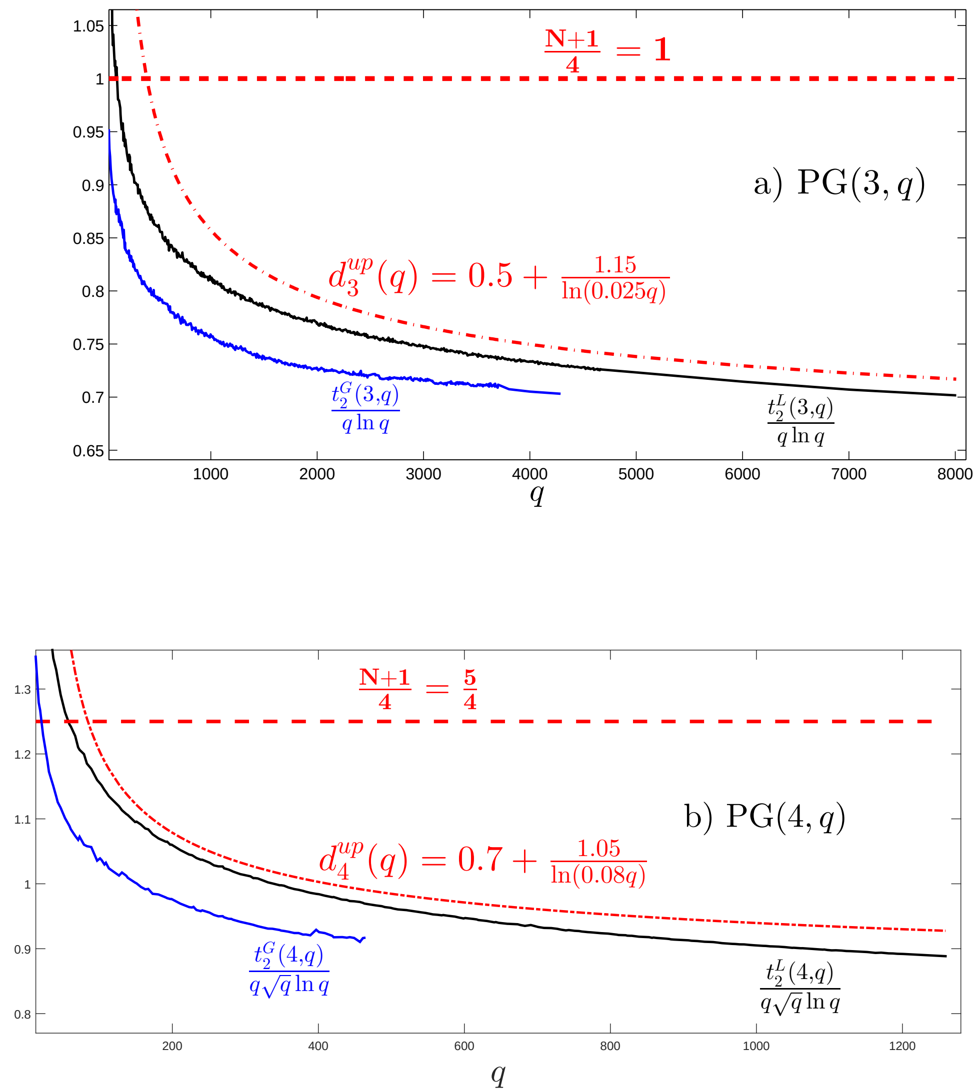
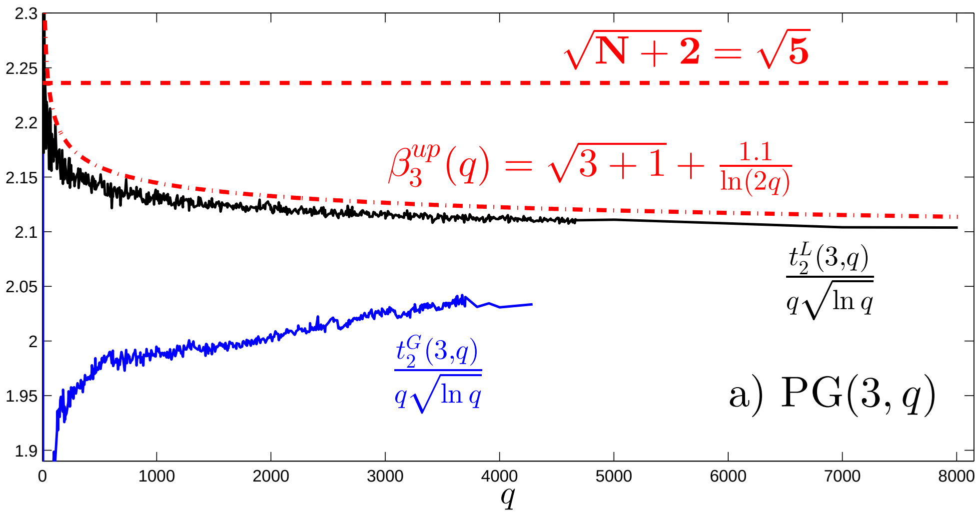


Figure 3: **Upper bounds** $\frac{t_2(N,q)}{q^{\frac{N-1}{2}}\ln q} < \frac{N+1}{4}$ (*dashed line* $y = \frac{N+1}{4}$) and $\frac{t_2(N,q)}{q^{\frac{N-1}{2}}\ln q} < d_N^{up}(q)$ (*top dashed-dotted curve*) **vs values** $\frac{t_2^L(N,q)}{q^{\frac{N-1}{2}}\ln q}$, $q \in L_N$ (*the 2-nd solid curve*) and $\frac{t_2^G(N,q)}{q^{\frac{N-1}{2}}\ln q}$, $q \in G_N$(*bottom solid curve*).
a) $N = 3$, $d_3^{up} = 0.5 + 1.15/\ln(0.025q)$; b) $N = 4$, $d_4^{up} = 0.7 + 1.05/\ln(0.08q)$



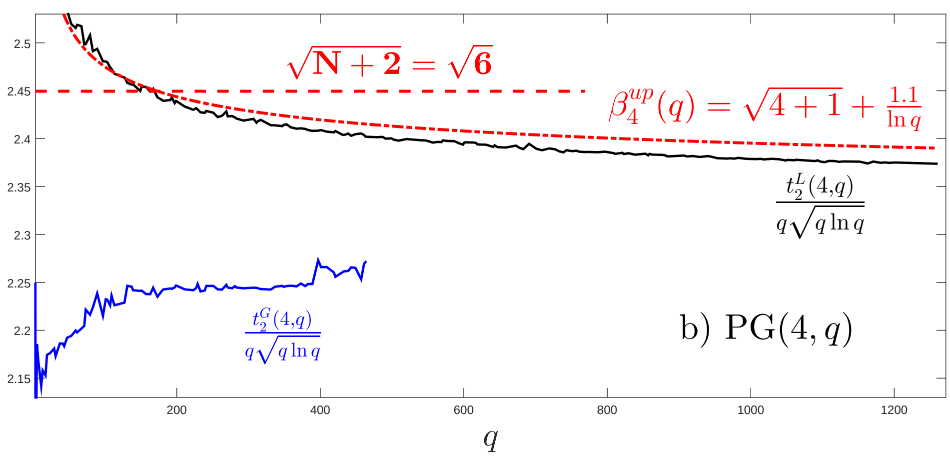


Figure 4: **Upper bounds** $\frac{t_2(N,q)}{q^{\frac{N-1}{2}}\sqrt{\ln q}} < \sqrt{N+2}$ (*dashed line* $y = \sqrt{N+2}$) and $\frac{t_2(N,q)}{q^{\frac{N-1}{2}}\sqrt{\ln q}} < \beta_N^{up}(q)$ (*top dashed-dotted curve*) **vs values** of $\frac{t_2^L(N,q)}{q^{\frac{N-1}{2}}\sqrt{\ln q}}$, $q \in L_N$ (*the 2-nd solid curve*) and $\frac{t_2^G(N,q)}{q^{\frac{N-1}{2}}\sqrt{\ln q}}$, $q \in G_N$ (*bottom solid curve*). a) $N = 3$, $\beta_3^{up} = \sqrt{3+1} + 1.1/\ln(2q)$; b) $N = 4$, $\beta_4^{up} = \sqrt{4+1} + 1.1/\ln q$