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Construction of some triple blocking sets in PG$(2, q)$ [1]

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Abstract. A b-set B in PG$(2, q)$, the projective plane over the field of q elements, is called a (b, m)-*blocking set* if every line meets B in at least m points and some line meets B in exactly m points. B is called a *triple blocking set* if $m = 3$. When B contains a line for $m = 3$, it is known that $b = |B| \geq 4q$ if q is odd and that $b \geq 4q - 1$ if q is even. We show that there exist at least six $(4q, 3)$-blocking sets for odd $q \geq 7$ and three $(4q - 1, 3)$-blocking sets for even $q \geq 8$ which are projectively inequivalent.

1 Introduction

A b-set B in PG$(2, q)$ is called a (b, m)-*blocking set* if every line meets B in at least m points and some line meets B in exactly m points. B is called a *triple blocking set* if $m = 3$ [1]. When B contains a line for $m = 3$, it is known that $b = |B| \geq 4q$ if q is odd and that $b = |B| \geq 4q - 1$ if q is even [5].

Lemma 1 (Example 2.3 in [7]). *Let B_0 be the set of points on the lines* [100], [010], [001], [111] *together with the points* $\mathbf{P}(-1, 1, 1)$, $\mathbf{P}(1, -1, 1)$. *Then, B_0 forms a $(4q - 1, 3)$-blocking set if q is even and a $(4q, 3)$-blocking set if q is odd, where $[abc]$ denotes the line $\{\mathbf{P}(x, y, z) \in PG(2, q) \mid ax + by + cz = 0\}$.*

In this paper, we construct new $(4q, 3)$-blocking sets for odd q and $(4q-1, 3)$-blocking sets for even q in PG$(2, q)$. A line l is called an i-line for B if $|B \cap l| = i$. We denote by b_i the number of i-lines for a given blocking set B.

Theorem 2. *For odd $q \geq 5$, let C be a conic in $\Sigma = \mathrm{PG}(2, q)$. For any three points P_1, P_2, P_3 in C, let l_i be the tangent of $\mathcal{C}$ through P_i and l_{ij} be*

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the secant of C through P_i and P_j, and let $P_{ij} = l_i \cap l_j$ for $1 \le i \le j \le 3$. Take any two points P and Q from the three points P_{12}, P_{23}, P_{13}, and let $B = C \cup l_{12} \cup l_{23} \cup l_{13} \cup \{P, Q\}$. Then, B is a $(4q, 3)$-blocking set with spectrum $(b_3, b_4, b_5, b_6) = (15, 10, 1, 5)$ for $q = 5$ and

$$(b_3, b_4, b_5, b_6, b_{q+1}) = (\frac{(q+5)(q-2)}{2}, 2q, \frac{(q-3)(q-4)}{2}, q-3, 3) \text{ for } q \ge 7.$$

Proof. Let $C = \{P_1, P_2, \ldots, P_{q+1}\}$ be a conic in Σ and let l be a line. If l contains none of P_1, P_2, P_3, then l meets $l_{12} \cup l_{23} \cup l_{13}$ at three points. Thus, $|l \cap B| \ge 3$. If l contains exactly one of P_1, P_2, P_3, say P', l meets $l_{12} \cup l_{23} \cup l_{13}$ at two points. Then, l is a secant or a tangent of C. If l is a secant of C, l meets C at P' and another point. So, $|l \cap B| \ge 3$. If l is a tangent of C, l is l_1, l_2 or l_3, and l contains at least one of the points P and Q. So, $|l \cap B| \ge 3$. If l contains two of P_1, P_2 and P_3, then l is l_{12}, l_{23} or l_{13}. Thus, B is a $(4q, 3)$-blocking set. Without loss of generality, we may take $P = P_{13}$ and $Q = P_{12}$. Assume $q \ge 7$. The $(q+1)$-lines for B are l_{12}, l_{23}, l_{13}. So, $b_{q+1} = 3$. The 6-lines are the secants through P or Q except $\langle P, P_2\rangle$ and $\langle Q, P_3\rangle$. Hence $b_6 = 2(\frac{q-1}{2} - 1) = q - 3$. For $q = 5$, the above $(q+1)$-lines are also 6-lines for B, and $b_6 = 5$. Now, assume $q \ge 5$. The 5-lines are the secants of C passing through none of P_1, P_2, P_3 except the 6-lines. So, $b_5 = \binom{q+1-3}{2} - b_6 = (q-3)(q-4)/2$. The 4-lines are the external lines of C through P or Q, the secants $\langle P, P_2\rangle$, $\langle Q, P_3\rangle$, the tangents at $P_4, P_5, \ldots, P_{q+1}$ and $\langle P, Q\rangle$. Hence, $b_4 = q - 1 + 2 + (q + 1 - 3) + 1 = 2q$. Finally, $b_3 = \theta_2 - b_4 - b_5 - b_6 - b_{q+1} = (q + 5)(q - 2)/2$. □

Theorem 3. *Under the conditions of Theorem 2 with $q \ge 7$, take $P = P_{13}$, $Q = P_{12}$ and a point Q' in l_2 with $Q' \notin \{Q, P_2, l_{13} \cap l_2\}$, and let $\ell = \langle P, Q'\rangle$. Then $B' = (B\backslash\{Q\}) \cup \{Q'\}$ is a $(4q, 3)$-blocking set with spectrum*

(1) $(b_3, b_4, b_5, b_6, b_{q+1}) = (\frac{(q+5)(q-2)}{2}, 2q, \frac{(q-3)(q-4)}{2}, q-3, 3)$ *if ℓ is a tangent,*

(2) $(b_3, b_4, b_5, b_6, b_7, b_{q+1}) = (\frac{(q+5)(q-2)}{2}, 2q-1, \frac{q^2-7q+18}{2}, q-6, 1, 3)$ *if ℓ is a secant,*

(3) $(b_3, b_4, b_5, b_6, b_{q+1}) = (\frac{q^2+3q-8}{2}, 2q-3, \frac{q^2-7q+18}{2}, q-4, 3)$ *if ℓ is an external line.*

Proof. Since ℓ is a tangent of C if and only if $Q' = P_{23}$, we get the spectrum (1) from Theorem 2 if ℓ is a tangent. As we have already seen in the proof of Theorem 2, the tanget $\langle Q, P\rangle$ and the secant $\langle Q, P_3\rangle$ are 4-lines, the other $(q-3)/2$ secants through Q are 6-lines and the $(q-1)/2$ external lines through Q are 4-lines for B. We denote by b_i and b'_i the number of i-lines for B and B', respectively. Note that $b'_{q+1} = b_{q+1}$, for $Q' \in l_2 \setminus \{P_2, l_{13} \cap l_2\}$.

If ℓ is a secant, then for B, the tangent ($\ne l_2$) through Q' is a 4-line, the secant ℓ is a 6-line, the secants $\langle Q', P_1\rangle$, $\langle Q', P_3\rangle$ are 3-lines, other $(q-7)/2$

secants on Q' are 5-lines and the $(q-1)/2$ external lines on Q' are 3-lines. Hence, $b'_3 = b_3 + 2 + (q-1)/2 - 2 - (q-1)/2 = b_3$, $b'_4 = b_4 - 2 - (q-1)/2 - 1 + 2 + (q-1)/2 = b_4 - 1$, $b'_5 = b_5 + (q-3)/2 + 1 - (q-7)/2 = b_5 + 3$, $b'_6 = b_6 - (q-3)/2 - 1 + (q-7)/2 = b_6 - 3$, $b'_7 = 1$.

If ℓ is an external line, then for B, the tangent ($\neq l_2$) through Q' is a 4-line, the secants $\langle Q', P_1\rangle$, $\langle Q', P_3\rangle$ are 3-lines, other $(q-5)/2$ secants on Q' are 5-lines, the external line ℓ is a 4-line and the $(q-3)/2$ external lines on Q' are 3-lines. Hence, $b'_3 = b_3 + 2 + (q-1)/2 - 2 - (q-3)/2 = b_3 + 1$, $b'_4 = b_4 - 2 - (q-1)/2 - 1 + 2 - 1 + (q-3)/2 = b_4 - 3$, $b'_5 = b_5 + (q-3)/2 + 1 - (q-5)/2 + 1 = b_5 + 3$, $b'_6 = b_6 - (q-3)/2 + (q-5)/2 = b_6 - 1$. □

We note that the construction of a $(4q, 3)$-blocking set with spectrum (1) or (3) in Theorem 3 is also valid for $q = 5$, but not for the spectrum (2) since ℓ is a secant if and only if $Q' = l_{13} \cap l_2$ when $q = 5$. See Corollary 7.5 in [8] for the next Lemma.

Lemma 4 ([8])**.** *In $PG(2, q)$ with $q \geq 4$, there is a unique conic through a 5-arc.*

We can get one more $(4q, 3)$-blocking set in $\mathrm{PG}(2, q)$ from the set B in Theorem 2 by two points exchange.

Theorem 5. *Let $q = p^h \geq 7$ with odd prime $p \neq 3$. Under the conditions of Theorem 2, let C be the conic $\{\mathbf{P}(1, a, a^2) \mid a \in \mathbb{F}_q\} \cup \{\mathbf{P}(0, 0, 1)\}$ and take $P_1 = \mathbf{P}(1, 1, 1)$, $P_2 = \mathbf{P}(0, 0, 1)$, $P_3 = \mathbf{P}(1, 0, 0)$, $P_4 = \mathbf{P}(1, 2^{-1}, 2^{-2})$, $P_5 = \mathbf{P}(1, 2, 2^2)$, $S = \langle P_1, P_4\rangle \cap \langle P_2, P_5\rangle$ and $T = \langle P_1, P_5\rangle \cap \langle P_3, P_4\rangle$. Then, $B_1 = (B\backslash\{P_4, P_5\}) \cup \{S, T\}$ is a $(4q, 3)$-blocking set, which is not projectively equivalent to any blocking set in Theorems 2 and 3.*

Proof. Note that $P_4 \neq P_5$ if $p \neq 3$ and that $S = \mathbf{P}(1, 2, 2 + 2^{-1})$, $T = \mathbf{P}(2 + 2^{-1}, 2, 1)$. Since $P = l_1 \cap l_3 = \mathbf{P}(1, 2^{-1}, 0)$ and $Q = l_1 \cap l_2 = \mathbf{P}(0, 1, 2)$, the lines $\langle P, P_2\rangle$ and $\langle Q, P_3\rangle$ are passing through P_4 and P_5, respectively. Let $B_1^- = B\backslash\{P_4, P_5\}$. Then, the 2-lines for B_1^- are $\langle P_1, P_4\rangle$, $\langle P_1, P_5\rangle$, $\langle P_2, P_5\rangle$ and $\langle P_3, P_4\rangle$. Hence, adding $S = \langle P_1, P_4\rangle \cap \langle P_2, P_5\rangle$ and $T = \langle P_1, P_5\rangle \cap \langle P_3, P_4\rangle$ to B_1^-, $B_1 = B_1^- \cup \{S, T\}$ forms a $(4q, 3)$-blocking set. It can be checked using a computer that B_1 has spectrum $(b_3, b_4, b_5, b_7, b_8) = (28, 18, 6, 2, 3)$ for $q = 7$, $(b_3, b_4, b_5, b_6, b_7, b_{12}) = (66, 38, 16, 8, 2, 3)$ for $q = 11$ and $(b_3, b_4, b_5, b_6, b_{14}) = (93, 44, 27, 16, 3)$ for $q = 13$. Hence, B_1 is not projectively equivalent to any blocking set in Theorems 2 and 3. Assume $q \geq 17$ and suppose B_1 contains a conic C'. Since $C \neq C'$, it follows from Lemma 4 that C' could contain at most 4 points from C, 6 points from $l_{12} \cup l_{13} \cup l_{23}$ and the other 4 points, totally at most 14 points from B_1, a contradiction. Thus, B_1 contains no conic for $q \geq 17$. On the other hand, the blocking sets in Theorem 2 and 3 contain a conic. Hence, the blocking set B_1 is not projectively equivalent to any blocking set in the previous theorems. □

Remark 6. (1) *Assume $q = 5$ in Theorem 5. It is known that there exist two inequivalent (20,3)-blocking sets (equivalently, (11,3)-arcs) in $PG(2,5)$, see also Table 12.5 in [8]. The (20,3)-blocking sets have spectrum*

(a) $(b_3, b_4, b_5, b_6) = (15, 10, 1, 5)$ *or*

(b) $(b_3, b_4, b_5, b_6) = (16, 7, 4, 4)$.

There are four 6-lines l_{12}, l_{13}, l_{23} and $\langle S, T\rangle$ for the blocking set B_1 in Theorem 5 when $q = 5$. So, B_1 has spectrum (b). Hence, B_1 is projectively equivalent to the blocking set in Theorem 3 (3).
(2) *When $q = 7$, the line $\langle P, S\rangle$ in the proof of Theorem 5 is a secant of C. On the other hand, when q is 13, $\langle P, S\rangle$ is an external line of C. Thus, the line $\langle P, S\rangle$ could form a tangent, a secant or an external line of C up to q. That is why we could not determine the spectrum of the $(4q, 3)$-blocking set in Theorem 5.*

Next, we determine the spectrum of the arc B_0 in Lemma 1 for odd q to find one more inequivalent arc.

Theorem 7. *For odd $q \geq 5$, let $B = l_1 \cup l_2 \cup l_3 \cup l_4 \cup \{P_1, P_2\}$, consisting of the lines $l_1 = [100]$, $l_2 = [010]$, $l_3 = [001]$, $l_4 = [111]$ and the points $P_1 = \mathbf{P}(-1, 1, 1)$, $P_2 = \mathbf{P}(1, -1, 1)$. Then, B forms a $(4q, 3)$-blocking set with spectrum $(b_3, b_4, b_5, b_{q+1}) = (6q - 14, q^2 - 7q + 17, 2q - 6, 4)$.*

Proof. Note that no three of the lines l_1, l_2, l_3, l_4 are concurrent. Let $\mathcal{Q} = \{Q_{ij} = l_i \cap l_j \mid 1 \leq i < j \leq 4\}$, $r_1 = \langle Q_{14}, Q_{23}\rangle$, $r_2 = \langle Q_{13}, Q_{24}\rangle$ and $r_3 = \langle Q_{12}, Q_{34}\rangle$. Then, P_1 and P_2 are equal to $r_2 \cap r_3$ and $r_1 \cap r_3$, respectively. Hence, $r_3 = \langle P_1, P_2\rangle$ is a 4-line. Let l be a line. l meets $\bigcup_{i=1}^4 l_i$ at two, three or four points. When $|l \cap (\bigcup_{i=1}^4 l_i)| = 2$, l is r_1, r_2 or r_3. So, l contains P_1 or P_2. Thus, B is a $(4q, 3)$-blocking set. Now, the $(q + 1)$-lines for B are $l_1, \ldots, l_4$, and $b_{q+1} = 4$. The 5-lines for B are the lines containing one of P_1, P_2 but none of $\mathcal{Q}$. Hence, $a_5 = 2(q + 1 - 4)$. The 3-lines for B are the lines through one of two points Q_{12}, Q_{34} containing no other point of $\mathcal{Q}$, the lines through one point ($\neq Q_{12}, Q_{34}$) of $\mathcal{Q}$ containing none of $\{P_1, P_2\}$, and two more lines r_1, r_2. Thus, $b_3 = 2(q + 1 - 3) + 4(q + 1 - 4) + 2 = 6q - 14$. Finally, $b_4 = \theta_2 - b_{q+1} - b_5 - b_3 = q^2 - 7q + 17$. □

Theorem 8. *Under the conditions of Theorem 7, let $P_3 = r_1 \cap r_2$. Take $P_2' \in r_1 \backslash \{P_2, P_3, Q_{14}, Q_{23}\}$ and let $B' = (B \backslash \{P_2\}) \cup \{P_2'\}$. Then, B' is a $(4q, 3)$-blocking set with spectrum $(b_3, b_4, b_5, b_6) = (15, 10, 1, 5)$ for $q = 5$ and $(b_3, b_4, b_5, b_6, b_{q+1}) = (6q - 15, q^2 - 7q + 20, 2q - 9, 1, 4)$ for $q \geq 7$.*

Proof. Since the 3-line for B through P_2 is r_1 only, B' forms a $(4q, 3)$-blocking set. The lines through P_2 for K except $r_1 = \langle P_2, P_2'\rangle$ are three 4-lines $\langle P_2, Q_{13}\rangle$, $\langle P_2, Q_{24}\rangle$, $\langle P_1, P_2\rangle$ and $(q - 3)$ 5-lines. On the other hand, the lines through

P'_2 for K other than r_1 are four 3-lines $\langle P'_2, Q_{ij}\rangle$ with $Q_{ij} \in \mathcal{Q} \setminus r_1$, one 5-line $\langle P'_2, P_1\rangle$ and $(q-5)$ 4-lines. Hence, $b'_3 = b_3+3-4$, $b'_4 = b_4-3+(q-3)+4-(q-5)$, $b'_5 = b_5 - (q-3) - 1 + (q-5)$, $b'_6 = 1$ (or $b'_6 = 1+4 = 5$ for $q = 5$), where b_i and b'_i are the number of i-lines for B and B', respectively. Now, our assertion follows from Theorem 7. □

An n-set in PG$(2, q)$ at most r points of which are collinear is called an (n, r)-*arc* in PG$(2, q)$, see [1], [2], [3]. For an n-set K and its complement $B = \Sigma\backslash K$ in $\Sigma =$PG$(2, q)$, K is an (n, r)-arc if and only if B is a $(\theta_2 - n, \theta_1 - r)$-blocking set. From the above theorems, we get the following.

Corollary 9. *There exist at least six projectively inequivalent $(q^2-3q+1, q-2)$-arcs in PG$(2, q)$ for odd $q \geq 7$.*

Finally, we consider the case q is even. Assume $q \geq 4$. Then, it is known that a $(b, 3)$-blocking set B containing a line satisfies $b \geq 4q-1$ [6]. The set B_0 for even q in Lemma 1 is such a $(4q-1, 3)$-blocking set with spectrum

$$(b_3, b_4, b_5, b_{q+1}) = (6q-9, q^2-6q+8, q-2, 4).$$

When $q = 4$, the complement of a $(4q-1, 3)$-blocking set is a 6-arc (a hyperoval). So, assume $q \geq 8$. We can construct two more $(4q-1, 3)$-blocking sets as follows.

Theorem 10. *For even $q \geq 8$, let C be a conic in $\Sigma =$ PG$(2, q)$ with nucleus N. For any three points P_1, P_2, P_3 in $C\cup\{N\}$ with $P_1, P_2 \in C$, let $l_{ij} = \langle P_i, P_j\rangle$ for $1 \leq i < j \leq 3$. Then,*

(1) *$B = C\cup l_{12}\cup l_{23}\cup l_{13}$ is a $(4q-1, 3)$-blocking set with spectrum $(b_3, b_5, b_{q+1}) = (\frac{(q+6)(q-1)}{2}, \frac{(q-1)(q-2)}{2}, 3)$ with $|Aut(B)| = 2(q-1)$ if $P_3 = N$,*

(2) *$B = C \cup l_{12} \cup l_{23} \cup l_{13} \cup \{N\}$ is a $(4q-1, 3)$-blocking set with spectrum $(b_3, b_5, b_{q+1}) = (\frac{(q+6)(q-1)}{2}, \frac{(q-1)(q-2)}{2}, 3)$ with $|Aut(B)| = 6$ if $P_3 \neq N$.*

The $(4q-1, 3)$-blocking sets in Theorem 10 were first found for $q = 8$, see [4].

Corollary 11. *There exist at least three projectively inequivalent $(4q-1, 3)$-blocking sets (equivalently, $(q^2-3q+2, q-2)$-arcs) in PG$(2, q)$ for even $q \geq 8$.*

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