

Colength of $*$ -Polynomial Identities of Simple $*$ -Algebras

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Abstract. We compute the sequence of colengths in the cocharacters of the $*$ -polynomial identities of the $*$ -simple algebra $M_2(F) \oplus M_2(F)^{\text{op}}$, $\text{char}(F) = 0$.

1 Introduction

Let A be an algebra over a field F . A function $*$: $A \rightarrow A$ is said to be an *involution* if $*$ is an automorphism of the additive group of A such that $(ab)^* = b^*a^*$ and $(a^*)^* = a$ for all $a, b \in A$. An example of such a map is the transpose in the algebra $M_n(F)$ of $n \times n$ matrices over the field F .

In particular in the case of a unitary algebra $(A, *)$ with involution $*$, there are two possibilities: the restriction of $*$ on F is the identical map (is called the involution of the first kind), otherwise $*$ is referred to be an involution of the second type. In the theory of algebras with polynomial identities one tries to take into account the existence of the additional structure in the algebra. In particular, for algebras with involution we consider the so called $*$ -polynomial identities. In this case usually one studies algebras with involution of the first kind only.

The algebra $(A, *)$ is said to be *$*$ -simple* if $A^2 \neq 0$ and it has no nontrivial $*$ -invariant ideals.

The *opposite algebra* of A , denoted by A^{op} , is the algebra that has the same elements as A , the same addition as A , and multiplication given by $a \circ b = ba$, where ba is a product in A . It is easy to check that $(A^{\text{op}})^{\text{op}} = A$, $A \cong B$ if and only if $A^{\text{op}} \cong B^{\text{op}}$. In this case the algebra $A \oplus A^{\text{op}}$ has the exchange involution defined by $(a, b)^* = (b, a)$.

The description of $*$ -simple algebras is given in [8, Proposition 2.13.24].

Theorem 1.1 *Let A be a $*$ -simple finite dimensional associative algebra over an algebraically closed field. Then either A is simple as an algebra or A is of the form $A = B \oplus B^{\text{op}}$, where B is a simple algebra.*

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Over an algebraically closed field F the $*$ -simple finite dimension algebras are: $(M_n(F), t)$ with transpose, $(M_n(F), s)$ symplectic involution (for even n), respectively and $M_n(F) \oplus M_n(F)^{\text{op}}$.

Drensky and Giambruno [4] have obtained the exact values of the cocharacters, codimensions and the Hilbert series of the polynomial identities of the $*$ -simple algebras $(M_2(F), t)$ and $(M_2(F), s)$.

The subject of our study is the algebra $M_2(F) \oplus M_2(F)^{\text{op}}$ with exchange involution corresponding to the second case in Theorem 1.1. We obtain the sequence of colengths of its $*$ -identities in the case when F is of characteristic zero.

2 Preliminaries

In this paper we consider the algebra $M_2 \oplus M_2^{\text{op}}$ and its $*$ identities.

Let A be an algebra with involution over a field F of characteristic 0. The free associative algebra with involution $F\langle X, * \rangle$ is the free associative algebra on the set of free generators $X \cup X^*$ where $X = \{x_1, x_2, \dots\}$ and $X^* = \{x_1^*, x_2^*, \dots\}$ and involution that extends the map $x_i \xrightarrow{*} x_i$ and $x_i^* \xrightarrow{*} x_i$. A polynomial $f(X, X^*) \in K\langle X, * \rangle$ is a $*$ -polynomial identity for the algebra $(A, *)$ if $f(a_1, \dots, a_n; a_1^*, \dots, a_n^*) = 0$ for all $a_i \in A$. We denote by $T(A, *)$ the ideal of all $*$ -polynomial identities of $(A, *)$. Instead, it is more convenient to change the variables and to assume that $y_i = \frac{1}{2}(x_i + x_i^*)$, $z_i = \frac{1}{2}(x_i - x_i^*)$ are the symmetric and skew variables, respectively. Then $F\langle X, * \rangle = F\langle Y, Z, * \rangle$. Let $Y_p = \{y_1, \dots, y_p\}$ be a set of symmetric variables $y_i \in F\langle Y, Z, * \rangle$, and $Z_q = \{z_1, \dots, z_q\}$ be a set of skew variables $z_i \in F\langle Y, Z, * \rangle$.

Let us denote the sets of symmetric and skew elements of A by $A^+ = \{a \in A \mid a^* = a\}$ and $A^- = \{a \in A \mid a^* = -a\}$, respectively. Consequently, $f(Y, Z) \in T(A, *)$ if and only if the polynomial $f(y_1, \dots, y_p, z_1, \dots, z_q)$ is such that $f(b_1, \dots, b_p, c_1, \dots, c_q) = 0$ for all $b_i \in A^+$, $i = 1, \dots, p$ and $c_j \in A^-$, $j = 1, \dots, q$.

The factor algebra $F(A, *) = F\langle Y, Z, * \rangle / T(A, *)$ is the relatively free algebra in the variety of algebras with involution generated by $(A, *)$. We denote by $F_{p,q}(A, *)$ the subalgebra of $F(A, *)$ generated by $Y_p = \{y_1, \dots, y_p\}$ and $Z_q = \{z_1, \dots, z_q\}$ and assume that by $F_m(A, *) = F_{m,m}(A, *)$.

The Hilbert series of $F_{p,q}(A, *)$ is defined as a formal power series

$$H(A, *, y_1, \dots, y_p, z_1, \dots, z_q) = \sum_{(a,b)} \dim F_{p,q}^{(a,b)} y_1^{a_1} \dots y_p^{a_p} z_1^{b_1} \dots z_q^{b_q}$$

or if we use the shorter notation $Y_m^a = (y_1^{a_1} \dots y_p^{a_p})$ and $Z_m^b = (z_1^{b_1} \dots z_q^{b_q})$ then

$$H(A, *, Y_p, Z_q) = \sum_{(a,b)} \dim F_{p,q}^{(a,b)} Y_p^a Z_q^b$$

For ordinary polynomial identities one of the most important numerical invariants of the polynomial identities of A is the S_n -cocharacter sequence. Similarly for $*$ -polynomial identities one considers the characters of the wreath product $\mathbb{Z}_2 \wr S_n$ [6]. Let us denote by $\chi_{\lambda, \mu}$ the irreducible $\mathbb{Z}_2 \wr S_n$ -character associated with the pair of partitions (λ, μ) . The $\mathbb{Z}_2 \wr S_n$ -module structure of the set of multilinear polynomials in Y and Z namely $P_n(A, *)$ and the $GL_m \times GL_m$ -module structure of $F_m(A, *)$ are related by the following results given by Giambruno.

Theorem 2.1 ([7, Theorems 1 and 2]) *If*

$$\chi_n(A, *) = \sum_{|\lambda|+|\mu|=n} m_{\lambda, \mu} \chi_{\lambda, \mu},$$

$$H(A, *, Y_m, Z_m) = \sum_{n \geq 0} \sum_{|\lambda|+|\mu|=n} b_{\lambda, \mu} S_\lambda(Y_m) S_\mu(Z_m),$$

then $m_{\lambda, \mu} = b_{\lambda, \mu}$ for all λ, μ , where $S_\lambda(Y_m)$ and $S_\mu(Z_m)$ are the Schur functions indexed by λ and μ , respectively.

By analogy with the proper (or commutators) polynomial identities for ordinary PI algebras, in the $*$ -case one considers the so-called \mathcal{Y} -proper polynomial identities. They are the $*$ -identities in which all symmetric variables participate in commutators only (see [4]).

By analogy ([6]), the corresponding relations for $*$ -polynomial identities of the Hilbert series of the relatively free algebra and its proper elements ($B_m(A, *)$) is

$$H(F_m(A, *), Y_p, Z_q) = H(B_m(A, *), Y_p, Z_q) \prod_{i=1}^m \frac{1}{1 - y_i}.$$

Studying the cocharacters of an algebra A , one considers also its sequence of colengths. In the $*$ -case this is

$$l_n(A, *) = \sum_{k=0}^n l_{k, n-k}(A, *) \quad \text{where} \quad l_{k, n-k}(A, *) = \sum_{\substack{\lambda \vdash k \\ \mu \vdash n-k}} m_{\lambda, \mu}, \quad n = 1, 2, \dots,$$

i.e. the sequence of lengths of the modules $P_n(A, *)$.

3 Colength of $A = M_2 \oplus M_2^{\text{op}}$

Drensky and Giambruno [5] have obtained the Hilbert series for the proper elements $B_{p,q}$ for algebra $M_2 \oplus M_2^{\text{op}}$, i.e

$$H(B_{p,q}, T_p, U_q) = \prod_{i=1}^p \frac{1}{1-t_i} \prod_{j=1}^q \frac{1}{(1-u_j)^2} \left(\sum_{n \geq 1} S_{(n,n)}(T_p, U_q) \right) - c(T_p, U_q),$$

$$\sum S_{(n,n)}(T_p, U_q) = \sum S_{(\lambda_1, \lambda_2)}(T_p) S_{(\mu_1, \mu_2)}(U_q), \quad (1)$$

where the summation runs on all (λ_1, λ_2) and (μ_1, μ_2) with $\lambda_1 + \mu_2 = \lambda_2 + \mu_1$ and the corrections $c(T_p, U_q)$ is

$$c(T_p, U_q) = \prod_{j=1}^p \frac{1}{1-u_j} \left(S_{(1^3)}(T_p, U_q) + \sum_{n \geq 1} S_{(n)}(T_p, U_q) \right).$$

The description of the multiplicities $m_{\lambda, \mu}$ is given in terms of the multiplicity series of the polynomial $f(T_p, U_q)$

$$M(f, T_p, U_q) = \sum_{\lambda \mu} m_{\lambda \mu} T_p^\lambda U_q^\mu.$$

Also we denote by \mathcal{Y}_T (and similarly \mathcal{Y}_U) the **Young operator** which sends the multiplicities series of $f(T_p, U_q)$ to the multiplicities series of $\prod_{i=1}^p \frac{1}{1-t_i} f(T_p, U_q)$, i.e.,

$$\mathcal{Y}_T(M(f(T_p, U_q))) = M\left(\prod_{i=1}^p \frac{1}{1-t_i} f(T_p, U_q)\right)$$

Since

$$H\left(\sum_{\lambda_1 + \mu_2 = \lambda_2 + \mu_1} S_{(\lambda_1, \lambda_2)}(T_p) S_{(\mu_1, \mu_2)}(U_q)\right) = \frac{1}{(1-t_1 t_2)(1-u_1 u_2)(1-t_1 u_1)}$$

we obtain that

Theorem 3.1 *The multiplicity series of proper identities of the algebra $M_2 \oplus M_2^{\text{op}}$ is obtained by applying two times Young operator with respect to U and ones to T , i.e.,*

$$M(B(M_2 \oplus M_2^{\text{op}}), T_p, U_q) = \mathcal{Y}_T \mathcal{Y}_U^2 \left(\frac{1}{(1-t_1 t_2)(1-u_1 u_2)(1-t_1 u_1)} \right)$$

Let $f_1 = \mathcal{Y}_T \mathcal{Y}_U^2 \left(\frac{1}{(1-t_1 t_2)(1-u_1 u_2)(1-t_1 u_1)} \right)$. After applying Young operator (see [3]) the number of variables increases by one. Hence in our case variables become t_1, t_2, t_3 and u_1, u_2, u_3, u_4 . To find the colength we have to substitute $t_1 = t_2 = t_3 = t$ and $u_1 = u_2 = u_3 = u_4 = u$. Then we obtain:

$$f_1 = \frac{t^5 u^9 + t^4 u^8 - t^4 u^4 + t^3 u^7 - t^3 u^4 - u^3 t^3 - t^2 u^6 - t^2 u^5 + t^2 u^2 - t u^5 + t u + 1}{(1+u^2)(1-t)^2(1-t^3)(1-t^2 u)(1+u)^3(1+t)(1-tu)^2(1+tu)^2(1-tu^2)^2(1-u^3)^2(1-u)^5} \quad (2)$$

$$\begin{aligned} f_2 &= \prod_{j=1}^p \frac{1}{1-u_j} S_{(1,1,1)}(T_p, U_q) = \prod_{j=1}^p \frac{1}{1-u_j} \sum_{k=1}^3 S_{(1^k)}(T_p) S_{(1^{3-k})}(U_q) \\ &= \prod_{j=1}^p \frac{1}{1-u_j} \left(S_{(0)}(T_3) S_{(1^3)}(U_4) + S_{(1)}(T_3) S_{(1^2)}(U_4) + S_{(1^2)}(T_3) S_{(1)}(U_4) + S_{(1^3)}(T_3) S_{(0)}(U_4) \right) \\ &= \frac{u^3 + tu^2 + t^2 u + t^3}{(1-u)^4} \end{aligned} \quad (3)$$

$$\begin{aligned} f_3 &= \prod_{j=1}^p \frac{1}{1-u_j} S_{(n)}(T_p, U_q) = \prod_{j=1}^p \frac{1}{1-u_j} \sum_{n \geq 1} S_{(k)} \times S_{(n-k)} \\ &= \frac{1}{(1-u)^4} \left(\frac{1}{(1-t)^3(1-u)^4} - 1 \right) \end{aligned} \quad (4)$$

4 Results

The main result of our paper is the following

Theorem 4.1 *The colength series of the $*$ -identities of $M_2 \oplus M_2^{op}$ is*

$$l(t, u) = M(H(B(M_2 \oplus M_2^{op}), t, t, t, u, u, u, u)) = f_1 - f_2 - f_3,$$

where f_1, f_2 and f_3 are given by equations (2), (3) and (4).

We use denote $M(f, T, U) = \sum_n \sum_k \sum_{\lambda, \mu} m_{\lambda\mu} T^\lambda, U^\mu$ and when one substitutes $t_i = t, i = 1, \dots, p$ and $u_j = u$ for $j = 1, \dots, q$ then obtains $M(f, t, u) = \sum_n \sum_k \left(\sum_{\lambda, \mu} m_{\lambda\mu} \right) t^k u^{n-k} = \sum_n \sum_k l_{k, n-k} t^k u^{n-k}$. This is the way to get the colength $l_{k, n-k} = \sum_{\lambda, \mu} m_{\lambda\mu}$.

We use Maple for symbolic computation, some technique making the rational function into partial fractions and we obtain the expression for $l_{k, n-k}$ in terms of n and k . The colength is too large to be included here. We are still looking for better form of it.

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