Steiner triple systems $S(2^m - 1, 3, 2)$ of 2-rank $r \leq 2^m - m + 1$: construction and properties

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Outline

A Steiner system $S(v,k,t)$ is a pair (X,B), X is a v-set (i.e. $|X|=v$) and B – the collection of k-subsets of X (called blocks) such that every t-subset (of t elements) of X is contained in exactly one block of B.

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Present a Steiner system $S(v, 3, 2)$ ($S(v, 4, 3)$) by the binary incidence matrix (rather a set of rows). It is a binary constant weight code C of length v, blocks of B are codewords.

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Now, we enumerate $S(v,3,2)$, where $v=2^m-1$, of rank 2^m-m+1 (min + 2).

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Let $J(v) = \{1, \dots, v\}$ be the coordinate set of S_v. Set $u = (v-3)/4$. Define the subsets J_i of $J(v)$:

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$$J(v) = J_1 \cup \cdots \cup J_u \cup J_{u+1}.$$

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For $x\in E^u$ with $\mathrm{supp}(\mathbf{x})=\{j_1,j_2,j_3\}$, we define a $(4u,3,4,16)$-code

$$C(L;\mathbf{x})=C(L;j_1,j_2,j_3)=\{(\mathbf{c}_1,\ldots,\mathbf{c}_u):\ (\mathbf{c}_{j_1},\mathbf{c}_{j_2},\mathbf{c}_{j_3})\in C(L)\},$$

and $\mathbf{c}_i=(0000)$ if $i\neq j_1,j_2,j_3$ (i.e. insert 3 blocks into u blocks).

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- $S^{(2,1)} = S^{(2,1)}_{v-2} \cup S^{(2,1)}_{v-1} \cup S^{(2,1)}_{v}$ is the set of words $\{\mathbf{c}\}$, $\mathrm{supp}(\mathbf{c}) = \{j_1,j_2,j_3\}$, $j_1,j_2 \in J_i$, and $j_3 \in J_{u+1}\}$. The set $S^{(2,1)}_{v-2}$:

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- $S^{(3)} = \{\mathbf{c}=(0\dots0111) : \operatorname{supp}(\mathbf{c}) = J_{u+1}\}$.

Theorem 1.

Let $S_u = S(u,3,2)$ *be a Steiner system and* $\mathbf{c}^{(s)}$, $s=1,2,\ldots,k$ *its words,* $k=u(u-1)/6$. *Let* $S^{(1,1,1)}$, $S^{(2,1)}$ *and* $S^{(3)}$ *be the sets, obtained by our construction, based on the families of* $(3,2,16)_4$*-codes* $L_1, L_2, \ldots, L_k$ *and the constant weight* $(4,2,4,2)$*-codes* $V(1)$, $V(2)$ *and* $V(3)$. *Set*

$$S \;=\; S^{(1,1,1)} \cup S^{(2,1)} \cup S^{(3)}.$$

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Then, for any choice of the codes $L_1, L_2, \ldots, L_k$, *the set* S *is the Steiner triple system* $S_v = S(v,3,2)$ *of order* $v = 4\,u+3$ *with rank*

$$v - (u - \mathrm{rk}(S_u)) - 2 \;\le\; \mathrm{rk}(S_v) \;\le\; v - (u - \mathrm{rk}(S_u)).$$

A system $S_u = S(u, 3, 2)$ of order $u = 2^l - 1$ is called boolean if its rank is $u - l$, i.e. it is formed by the codewords of weight 3 of the linear Hamming code of length u.

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Suppose $S_v = S(v, 3, 2)$ *is a Steiner system of order* $v = 2^m - 1 = 4u + 3$. *Suppose that its rank not greater than* $v - m + 2$.

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Suppose $S_v = S(v,3,2)$ is a Steiner system of order $v = 2^m - 1 = 4u + 3$. Suppose that its rank not greater than $v - m + 2$.

Then this system S_v is obtained from the boolean Steiner triple system $S_u = S(u,3,2)$ of order $u = 2^{m-2} - 1$ using our construction, described above.

Theorem 3.

The following is true:

- *Let* $m \geq 4$ *and* $v = 2^m - 1 \geq 15$. *Set* $u = (v-3)/4$ *and* $k = u(u-1)/6$. *Then, the number* M_v *of different Steiner triple systems* $S(v,3,2)$ *of order* v, *whose rank is not greater than* $v-m+2$, *and the fixed dual code* $\mathcal{A}_m$, *is equal to*

$$M_v = \left(2^6 \cdot 3^2\right)^k \times (6)^u\,, \quad k \ = \ u(u-1)/6.$$

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$$M_v = \left(2^6 \cdot 3^2\right)^k \times (6)^u \;, \quad k \;=\; u(u-1)/6.$$

- *The overall number* $M_v^{(o)}$ *of different Steiner triple systems* $S(v,3,2)$, *whose rank* $\le v - m + 2$, *is equal to*

$$M_v^{(o)} = \frac{v! \cdot \left(2^6 \cdot 3^2\right)^k \cdot (6)^u}{(u(u-1)(u-2)\cdots(u+1)/2) \cdot (4!)^u \cdot 3!}.$$

A system $S(v,3,2)$ of order $v = 2^m - 1$ is called *Hamming*, if it can be embedded into a binary non-linear perfect $(v,3,2^{v-m})$-code (denoted by H_v), i.e. if it is the set of words of weight 3 of the code H_v, which contains the zero codeword.

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Theorem 4.

Any Steiner triple system $S_v = S(v, 3, 2)$ *of order* $v = 2^m - 1$ *and rank* $\mathrm{rk}(S_v) \leq 2^m - m + 1$ *is a Hamming system.*