On the Error Exponent of Low-Complexity Decoded LDPC Codes with Special Construction

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Outline

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2. LDPC codes with special construction
3. Decoding algorithm
4. Main result
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Gallager's LDPC codes

Parity-check matrix of Gallager's LDPC code (G-LDPC code)

$$\mathbf{H_2} = \begin{pmatrix} \pi_1(\mathbf{H}_{b_0}) \\ \pi_2(\mathbf{H}_{b_0}) \\ \vdots \\ \pi_\ell(\mathbf{H}_{b_0}) \end{pmatrix}_{\ell b_0 \times b_0 n_0}$$

where

$$\mathbf{H}_{b_0} = \begin{pmatrix} \mathbf{H_0} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{H_0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{H_0} \end{pmatrix}_{b_0 \times b_0 n_0}$$

1. $\mathbf{H_0}$ is $1 \times n_0$ parity-check matrix of single parity-check code.
2. ℓ random column permutations of $\mathbf{H}_{b_0}$ form layers of $\mathbf{H_2}$.
3. Code rate is $R_2 \geqslant 1 - \ell/n_0$.

Ensemble of Gallager's LDPC codes

Definition

For a given constituent code with parity-check matrix $\mathbf{H}_0$, the elements of the ensemble $\mathscr{E}_G\left(n_0, \ell, b_0\right)$ are obtained by sampling independently the permutations π_l , $l = 1, 2, ..., \ell$, which are equiprobable.

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Remark

It is known[a] *that in ensemble* $\mathscr{E}_G(n_0, \ell, b_0)$ *of G-LDPC codes such code exists which can correct any error pattern with weight up to* $\lfloor \omega_t n \rfloor$ *while decoding with majority algorithm* $\mathscr{A}_M$ *with complexity* $\mathcal{O}(n \log n)$.

[a] P. S. Rybin and V. V. Zyablov, Asymptotic estimation of error fraction corrected by binary LDPC code, *2011 IEEE International Symposium on Information Theory Proceedings (ISIT)*, 2011, 351 – 355.

LDPC codes with special construction

Parity-check matrix of LDPC code with special construction:

$$\mathbf{H} = \begin{pmatrix} \mathbf{H}_2 \\ \pi\left(\mathbf{H}_{b_1}\right) \end{pmatrix}_{(\ell b_0 m_0 + b_1 m_1) \times b_0 n_0}$$

where

$$\mathbf{H}_{b_1} = \begin{pmatrix} \underbrace{\begin{matrix} \mathbf{H}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_1 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{H}_1 \end{matrix}}_{b_1} \end{pmatrix}_{b_1 m_1 \times b_1 n_1}$$

- $\mathbf{H}_1$ – parity-check matrix of “best” linear code.
- $\mathbf{H}_2$ – parity-check matrix of G-LDPC code from $\mathscr{E}_G\left(n_0, \ell, b_0\right)$.
- $R \geqslant R_1 + R_2 - 1$.

Ensemble of LDPC codes with special construction

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Definition

For a given linear code with parity-check matrix $\mathbf{H}_1$, the elements of the ensemble $\mathscr{E}_L\left(n_0, \ell, b_0, n_1, 1, b_1\right)$ are obtained by sampling independently the parity-check matrix $\mathbf{H}_2$ from $\mathscr{E}_G\left(n_0, \ell, b_0\right)$ and permutation π.

Decoding algorithm description

Decoding algorithm $\mathscr{A}_C$ consists of the following two steps:

1. Received sequence is decoded with well known maximum likehood algorithm separately by b_1 linear codes with parity-check matrix $\mathbf{H}_1$ from $\ell + 1$ layer of $\mathbf{H}$;

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2. Tentative sequence is decoded with well known majority decoding algorithm $\mathscr{A}_M$ by G-LDPC code with parity-check matrix $\mathbf{H}_2$.

Error exponent

- Investigating error probability P under decoding algorithm $\mathscr{A}_C$ of LG-LDPC code we considered memoryless BSC with BER p.
- Estimation on error probability P we wrote in the following way:

$$P \leqslant \exp\left\{-nE\left(R_1, n_1, \omega_t, p\right)\right\},$$

where $E\left(R_1, n_1, \omega_t, p\right)$ is error exponent.

Main result

If $R \to \mathscr{C}$ such, that $R_1 < \mathscr{C}$ and $R_2 < 1$, then $\exists n_1$, that $E\left(R_1, n_1, \omega_t, p\right) > 0$, if $\omega_t > 0$ for $\forall R_2 < 1$.

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Existence conditions:

- Let in the ensemble $\mathscr{E}_G\left(n_0, \ell, b_0\right)$ of G-LDPC codes such code with code rate R_2 exists, which can correct any error pattern of weight up to $\lfloor \omega_t n \rfloor$ while decoding with majority algorithm $\mathscr{A}_M$.
- Let the such linear code exists, which has code length n_1, code rate R_1 and error exponent of this code under maximum likehood decoding is lower-bounded with $E_0\left(R_1, p\right)$.

Theorem

If above conditions are fulfilled then in the ensemble $\mathscr{E}_L\left(n_0, \ell, b_0, n_1, 1, b_1\right)$ of LG-LDPC codes such code exists, which has the code length n:

$$n = n_0 b_0 = n_1 b_1,$$

code rate R:

$$R \geqslant R_1 + R_2 - 1$$

and error exponent of this code over memoryless BSC with BER p under decoding algorithm $\mathscr{A}_C$ with complexity $\mathcal{O}\left(n \log n\right)$ is lower-bounded with:

$$E\left(R_1, n_1, \omega_t, p\right) = \min_{\omega_t \leq \beta \leq \beta_0} \left\{ \beta E_0\left(R_1, p\right) + E_2\left(\beta, \omega_t, p\right) - \frac{1}{n_1} H\left(\beta\right) \right\}.$$

Notations

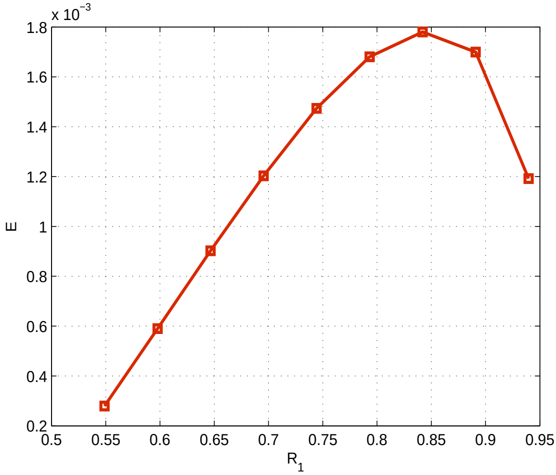
- $\beta_0 = \min\left(\frac{\omega_t}{2p}, 1\right);$
- $H(\beta) = -\beta \ln \beta - (1-\beta)\ln(1-\beta)$ – entropy function;
- $E_2(\beta, \omega_t, p)$ is given by:

$$E_2(\beta, \omega_t, p) = \frac{1}{2}\left(\omega_t \ln \frac{\omega_t}{p} + (2\beta - \omega_t)\ln \frac{2\beta - \omega_t}{1-p}\right) - \beta \ln(2\beta);$$

- n_1 satisfies the following conditions:

$$\frac{-\ln \beta_0}{E_0(R_1, p)} \leq n_1 \leq \frac{1}{R_1}\log_2\log_2(n).$$

Values of $E\left(R_1, n_1, \omega_t, p\right)$ according to R_1 of linear code and for fixed $R = 0.5$, $n_1 = 2000$ and $p = 10^{-3}$:



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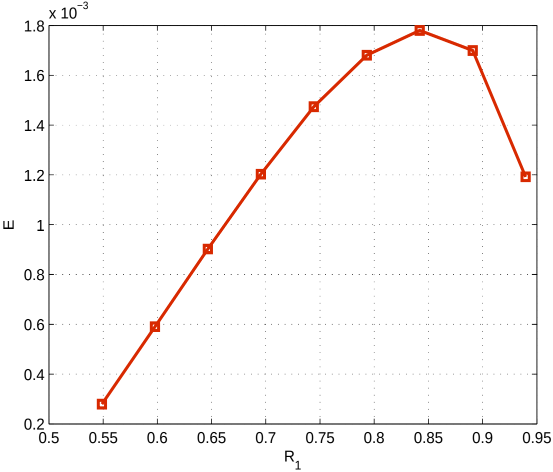
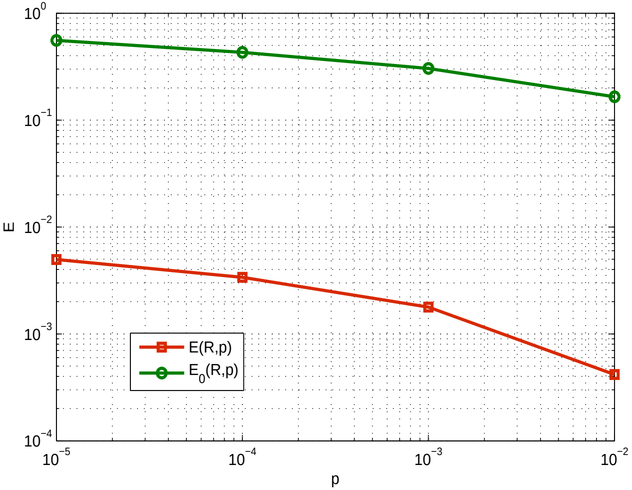
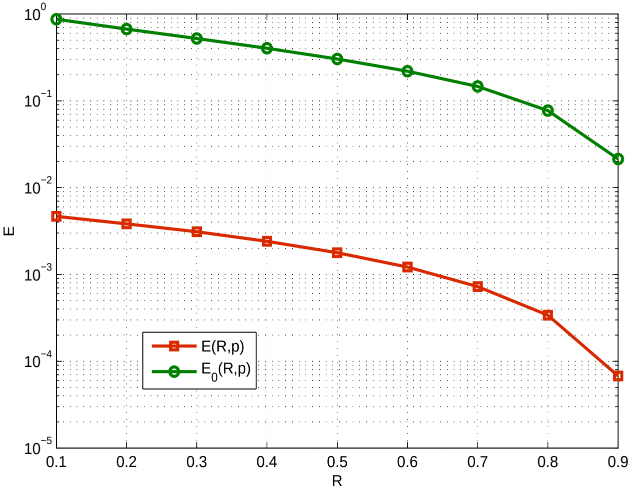


$$E\left(R, p\right) = \max_{R_1, R_2 : R_1 + R_2 - 1 = R} E\left(R_1, n_1, \omega_t, p\right).$$

Values of $E(R,p)$ and $E_0(R,p)$ according to p and for fixed $R = 0.5$:



Values of $E(R,p)$ and $E_0(R,p)$ according to R and for fixed $p = 10^{-3}$:



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- Decoding algorithm $\mathscr{A}_C$ with low complexity was developed.
- Lower-bound of error exponent for LG-LDPC codes under decoding algorithm $\mathscr{A}_C$ was obtained.
- It was proved, that for any code rate less than channel capacity such LG-LDPC code exists, that under decoding algorithm $\mathscr{A}_C$ with low-complexity $\mathcal{O}\left(n \log n\right)$ the error probability decreases exponentially.

Thank you for the attention!