On Klosterman sums over finite fields of characteristic 3

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Outline

We study the divisibility by 3^k of Klosterman sums $K(a)$ over finite fields of characteristic 3.

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We give a simple recurrent algorithm for finding the largest k, such that 3^k divides the Kloosterman sum $K(a)$.
This gives a simple description of zeros of such Kloosterman sums.

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$$K(a) = \sum_{x \in \mathbb{F}} \omega^{\mathrm{Tr}(x + a/x)}, \qquad (1)$$

where $\omega = \exp\{2\pi i/3\}$ is a primitive 3-th root of unity and

$$Tr(x) = x + x^3 + x^{3^2} + \cdots + x^{3^{m-1}}. \qquad (2)$$

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Recall that under x^{-i} we understand x^{3^m-1-i}, avoiding by this way a division into 0.

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In (Ahmadi O. & Granger R. [2011]) an efficient deterministic (recursive) algorithm was given proving divisibility of Klosterman sums by 3^k.

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We suggest also a recursive algorithm of finding the largest divisor of $K(a)$ of the type 3^k which does not need solving of cubic equation as in (Ahmadi O. & Granger R. [2011]), but only implementation of arithmetic operation in $\mathbb{F}$.
For the case when $m = g\,h$ we derive the exact connection between the divisibility by 3^k of $K(a)$ in $\mathbb{F}_{3^g}$, $a \in \mathbb{F}_{3^g}$, and the divisibility by $3^{k'}$ of $K(a)$ in $\mathbb{F}_{3^{gh}}$.

Our interest is the divisibility of such sums by the maximal possible number of type 3^k (i.e. 3^k divides $K(a)$, but 3^{k+1} does not divide $K(a)$; in addition, when $K(a) = 0$ we assume that 3^m divides $K(a)$, but 3^{m+1} does not divide).

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For a given $\mathbb{F}$ and any $a \in \mathbb{F}^*$ define the elliptic curve $E(a)$ as follows:

$$E(a) \;=\; \{(x, y) \in \mathbb{F} \times \mathbb{F} : \; y^2 = x^3 + x^2 - a\}. \qquad (3)$$

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The set of $\mathbb{F}$-rational points of the curve $E(a)$ over $\mathbb{F}$ forms a finite abelian group, which can be represented as a direct product of a cyclic subgroup $G(a)$ of order 3^t and a certain subgroup $H(a)$ of some order s (which is not multiple to 3):
$E(a) = G(a) \times H(a)$, such that

$$|E(a)| \;=\; 3^t \cdot s$$

for some integers $t \geq 2$ and $s \geq 1$ (Enge [1991]), where $s \not\equiv 0 \pmod 3$.

Moisio [2008] showed that

$$|E(a)| \; = \; 3^m + K(a), \tag{4}$$

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Since $|E(a)|$ is divisible by $|G(a)|$, which is equal to 3^t, then generator elements of $G(a)$ and only these elements are of order 3^t.

Let $Q = (\xi, *) \in E(a)$. Then the point $P = (x, *) \in E(a)$, such that $Q = 3P$ exists, if and only if the equation

$$x^9 - \xi x^6 + a(1-\xi)x^3 - a^2(a+\xi) \ = \ 0.$$

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The equation (5) is solvable in $\mathbb{F}$ if and only if

$$\mathrm{Tr}\left(\frac{a\sqrt{\xi^3+\xi^2-a}}{\xi^3}\right) \ = \ 0\ . \tag{6}$$

Since the point $(a^{1/3}, a^{1/3})$ belongs to $G(a)$ and has order 3, then solving the recursive equation

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with initial value $x_0 = a^{1/3}$, we obtain that the point $(x_i, *) \in G(a)$ for $i = 0, 1, \ldots, t-1$, and the point $(x_{t-1}, *)$ is a generator element of $G(a)$.

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Our purpose here is to generalize these results for finite fields of characteristic 3.

We begin with simple result. It is known (van der Geer - van der Vlugt [1991], Lisonek-Moisio [2011]) that 9 divides $K(a)$ if and only if $Tr(a) = 0$.

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$$x_1 = z^2(z+1)(z^2+1)(z-1)^4$$

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and, therefore, from condition (6), the following result holds.

Statement 1.

Let $a \in \mathbb{F}^*$ and $Tr(a) = 0$, i.e. a can be presented in the form: $a = z^{27} - z^9$. Then

$$x_0 = z^9 - z^3, \quad x_1 = z^2(z+1)(z^2+1)(z-1)^4,$$

and, therefore, $K(a)$ is divisible by 27, if and only if

$$\mathrm{Tr}\left(\frac{z^5(z-1)(z+1)^7}{(z^2+1)^3}\right) = 0, \qquad (8)$$

This condition (8) is less bulky than the corresponding condition from the paper (Ǵologlu-McGuire-Moloney [2011]), where it is proven that $K(a)$ is divisible by 27, if $\mathrm{Tr}(a) = 0$ and

$$2 \sum_{1 \leq i,j \leq m-1} a^{3^i+3^j} \; + \sum_{1 \leq i \neq j \neq k \leq m-1} a^{3^i+3^j+3^k} \; = \; 0.$$

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Similar to the case $p = 2$ (Bassalygo-Zinoviev [2011]), we give now also another algorithm to find the maximal divisor of $K(a)$ of the type 3^t, which does not require solving of the cubic equations (5), but only consequent implementation of arithmetic operations in $\mathbb{F}$.

Let $a \in \mathbb{F}^*$ be an arbitrary element and let $u_1, u_2, \ldots, u_\ell$ be a sequence of elements of $\mathbb{F}$, constructed according to the following recurrent relation (compare with (7)):

$$u_{i+1} = \frac{(u_i^3 - a)^3 + au_i^3}{(u_i^3 - a)^2}, \quad i = 1, 2, \ldots, \tag{9}$$

where $(u_1, *) \in E(a)$ and

$$Tr\left(\frac{a\sqrt{u_1^3 + u_1^2 - a}}{u_1^3}\right) \neq 0 \,. \tag{10}$$

Then the following result is valid.

Theorem 1.

Let $a \in \mathbb{F}^*$ and let $u_1, u_2, \ldots, u_\ell$ be a sequence of elements of $\mathbb{F}$, which satisfies the recurrent relation (9), where the element u_1 satisfies (10). Then there exists an integer $k \leq m$ such that one of the two following cases takes place:

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(i) either $u_k = a^{1/3}$, but all the previous u_i are not equal to $a^{1/3}$;
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(ii) or $u_{k+1} = u_{k+1+r}$ for a certain r and all u_i are different for $i < k + 1 + r$.
In the both cases the Kloosterman sum $K(a)$ is divisible by 3^k and is not divisible by 3^{k+1}.

Directly from Theorem 1 we obtain the following necessary and sufficient condition for an element $a \in \mathbb{F}^*$ to be a zero of the Kloosterman sum $K(a)$ (recall that the field $\mathbb{F}_q$ is of order $q = 3^m$).

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Corollary 2.

Let $a \in \mathbb{F}^*$ and $u_1, u_2, \ldots, u_\ell$ be the sequence of elements of $\mathbb{F}$, which satisfies the recurrent relation (9), where the element u_1 satisfies (10). Then $K(a) = 0$, if and only if $u_m = a^{1/3}$, and $u_i \neq a^{1/3}$ for all $1 \leq i \leq m-1$.

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Recall that

$$\mathrm{Tr}_{q^n \to q}(x) \;=\; x + x^q + x^{q^2} + \ldots + x^{q^{n-1}}, \quad x \in \mathbb{F}_{q^n},$$

and ω is a primitive 3-th root of unity. For any elements $a \in \mathbb{F}_q$ and $b \in \mathbb{F}_{q^n}$ define

$$e(a) \;=\; \omega^{\mathrm{Tr}(a)}, \quad e_n(b) \;=\; \omega^{\mathrm{Tr}(\mathrm{Tr}_{q^n \to q}(b))}.$$

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For a given $a \in \mathbb{F}_q^*$ it is possible to consider the following two Kloosterman sums:

$$K(a) \; = \; \sum_{x \in \mathbb{F}_q} e\left(x + \frac{a}{x}\right),$$

$$K_n(a) \; = \; \sum_{x \in \mathbb{F}_{q^n}} e_n\left(x + \frac{a}{x}\right).$$

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Theorem 3.

Let $n = 3^h \cdot s,\ \ n \geq 2,\ s \geq 1,$ *where* 3 *and* s *are mutually prime, and* $a \in \mathbb{F}_q^*$. *Then*

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From Theorem 3 we immediately obtain the following known result due to Lisonek and Moisio [2011].

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Let $n = 3^h \cdot s,\ \ n \geq 2,\ s \geq 1,$ where 3 and s are mutually prime, and $a \in \mathbb{F}_q^*$. Then

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From Theorem 3 we immediately obtain the following known result due to Lisonek and Moisio [2011].

Corollary 4.

Let $a \in \mathbb{F}_q^*$ and $n \geq 2$. Then $K_n(a)$ is not equal to zero.

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