Maximal Cliques in Graphs and Some New Upper Bounds for Constant-Weight Codes

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Abstract

Let $A(n,d,w)$ denote the maximum possible number of codewords in an (n,d,w) constant-weight binary code. In this paper we prove that $A(15,6,8) \leq 74$. There are unique codes attaining $A(12,6,8) = 9$, $A(13,6,8) = 18$ and $A(14,6,8) = 42$.

1 Introduction

An (n,d,w) constant-weight binary code is a set of binary vectors of length n, such that each vector contains w ones and $n-w$ zeros, and any two vectors differ in at least d positions. For given values of n, d and w, the maximum integer M such that an (n,d,w) code with M codewords exists is denoted by $A(n,d,w)$. Actually studying this function is a clique-finding problem. Construct the graph whose vertices represent binary vectors of length n. Join two vertices by an edge if and only if the Hamming distance between the vectors is at least d. Then what we are interested in is the quantity A(n,d,w), the size of the largest clique in this graph.

The results related to this function are summarized in the encyclopedic works of Brouwer and al. [3] and Agrel and al. [1]. In this paper we present some new upper bounds for $A(n,d,w)$ and some classification results.

Our approach is based on the observation that an (n,d,w) code C can be shortened to an $(n-1,d,w)$ code C' and an $(n-1,d,w-1)$ code C''. Conversely, if we want to construct an (n,d,w) code C, we only need to consider lengthening of an $(n-1, d, w)$ code C' or an $(n-1, d, w-1)$ code C''. There are two main problems. The first one is to construct all (n, d, w) codes with M vectors which contain C' (or C'') and the second problem is to find inequivalent between them. The construction of codes and backtrack search are considered in Section 2. Code equivalence is discussed in Section 3. We present our results in Section 4.

2 Construction of Codes

The following theorem derived from the Johnson bound:

Theorem 1 [3]. *Let C be an (n, d, w) code with M vectors. C contains $(n-1, d, w)$ codes with $M' = \lceil M\frac{n-w}{n}\rceil$ vectors and $(n-1, d, w-1)$ codes with $M'' = \lceil M\frac{w}{n}\rceil$ vectors.*

Let C_0 be an $(n-1, d, w)$ (or $(n-1, d, w-1)$) code. Our problem is how to find all the (n, d, w) codes which contain C_0 as a subcode. The search space will be only the vectors which are at a distance at least d from the code C_0. In the search, we will only have to care about the distance between the codewords. This is a clique-finding problem and we use a backtrack search with the punning techniques approach presented in [4].

3 Code equivalence

Let A and B be binary constant-weight codes. Two codes of the same size are equivalent if the codewords of the second one can be obtained by the codewords of the first one by a permutation of the coordinates. Any permutation of the coordinates of A which maps the codewords of A into codewords of the same code, is called an automorphism of A. The set of all automorphisms of A is a subgroup of the symmetric group S_n and we denote it by $Aut(A)$.

Let $N = \{1, 2, \ldots, n\}$ be the set of the coordinates of the codes and $R \subset N$. Let $Aut(A)_R \subset Aut(A)$ be the subgroup of all automorphisms of A which fix the coordinates from R. The group $Aut(A)_R$ partitions the set of coordinates into orbits. Two coordinates i and j are in the same orbit if there is an automorphism $\sigma \in Aut(A)_R$ such that $\sigma(i) = j$.

An invariant of a coordinate of the code A is a function $f : N \to Z[x]$ such that if i and j are in the same orbit with respect to $Aut(A)_R$ then

$f(i) = f(j)$ where $Z[x]$ is the ring of polynomials with integer coefficients.

We call a canonical map of A a function $C_A : N \to N$ such that if $A \cong B$ then $C_A(A) = C_B(B)$. We say that $C_A(A)$ is a canonical presentation of A.

For finding equivalence of codes we use an algorithm presented in [2]. This algorithm gives us a canonical presentation of codes. The main idea of this algorithm is to find all orbits in each step using invariants and then confirm that there are really orbits. As invariants we use different distances between coordinates.

4 Results

The results of the computer search can be found in Table 1. We started from the trivial result that there is a unique (11,6,8) code with two codewords. The number of codes with different parameters is given.

The total amount of CPU time needed for proving the nonexistence of (15,6,8) codes with 75 codewords was about 140 hours on a contemporary PC with the Windows 98 operating system. Most of this time, about 120 hours, we spent on the final step of extension of (14,6,8) codes with 35 codewords to (15,6,8) codes with 75 codewords.

Table 1
Number of Inequivalent Codes

n	M	# (n,6,8)	n	M	# (n,6,8)
11	2	1	14	35	84600
12	6	8	14	36	16270
12	7	2	14	37	2697
12	8	1	14	38	392
12	9	1	14	39	51
13	15	1189	14	40	7
13	16	22	14	41	1
13	17	2	14	42	1
13	18	1	15	75	0

Using Theorem 1 and trivial values $A(n, d, w) = A(n, d, n-w)$ we obtain the following new upper bounds:

Table 2: New Bounds

(n,d,w)	Lower bound	Upper bound	Upper bound in [1]
$(15, 6, 8)$	69	74	78
$(16, 6, 8)$	120	148	150
$(17, 6, 8)$	184	279	283
$(16, 6, 7)$	109	131	138
$(17, 6, 7)$	166	222	234

References

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