Classification of Griesmer Codes and Dual Transform

Iliya G. Bouyukliev
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
P.O.Box 323, 5000 Veliko Tarnovo, Bulgaria

Abstract

In this work, we consider a classification of infinite families of linear codes which achieve the Griesmer bound, using the projective dual transform. We investigate the correspondence between families of linear codes with given properties via dual transform.

Keywords: optimal linear codes, Griesmer bounds, classification, dual transform

In Memory of My Father

1 Introduction

The main aim of the paper is to develop a method for classification of infinite families of Griesmer codes. It is based on the fact that all codes from a family can be mapped to projective dual codes with fixed length and weights. Using it we classify many families of codes over fields with 2, 3 and 4 elements.

Let $\mathbb{F}_q^n$ be the n-dimensional vector space over the Galois field $\mathbb{F}_q$. The Hamming distance between two vectors of $\mathbb{F}_q^n$ is defined to be the number of coordinates in which they differ.

Definition 1. *A q-ary linear $[n,k,d]_q$-code is a k-dimensional linear subspace of $\mathbb{F}_q^n$ with minimum distance d.*

Definition 2. *A $k \times n$ matrix G having as rows the vectors of a basis of C is called a generator matrix of C.*

Definition 3. *Let C_1 and C_2 be two linear $[n,k]_q$ codes. They are said to be equivalent if the codewords of C_2 can be obtained from the codewords of C_1 via a sequence of transformations of the following types:*
(1) permutation on the set of coordinate positions;
(2) multiplication of the elements in a given position by a non-zero element of $\mathbb{F}_q$;
(3) application of a field automorphism to the elements in all coordinate positions.

Each of (1)-(3) preserves the Hamming weight and hence maps $[n,k,d]_q$ codes onto $[n,k,d]_q$ codes. Codes which are equivalent belong to the same equivalence class. To classify linear $[n,k,d]_q$ codes means to construct a representative of each equivalence class.

Let $n_q(k,d)$ be the smallest value of n, for which an $[n,k,d]_q$ code exists. Codes with parameters $[n_q(k,d),k,d]_q$ are called optimal.

There are two general problems in constructive coding theory:
- To construct optimal codes with given parameters.
- To classify optimal codes with given parameters.

A general lower bound on $n_q(k,d)$ has been proved by Griesmer and Varshamov for $q=2$ and by Solomon and Stiffler for any q:

Theorem 4. (Griesmer bound) [16],[32], [30] Let C be an $[n,k,d]_q$ code. Then

$$n_q(k,d) \geq g_q(k,d) = \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$.

Linear codes attaining the Griesmer bound, i.e. codes with parameters $[g_q(k,d),k,d]_q$, are called Griesmer codes. For given q and k, there is a constant D such that for all $d \geq D$ Griesmer $[g_q(k,d),k,d]$ codes exist [1], [12]. A natural question in this direction is what is the number of nonequivalent Griesmer codes when d increases. We prove that for given q and k and any $d_0 \leq q^{k-1}$ there is a constant L such that for $t \geq L$ the number of the non-equivalent Griesmer codes with minimum distance $d_0 + tq^{k-1}$, is a constant. Using projective dual transform, we develop a general approach for classification of infinite families of Griesmer codes with parameters $[g_q(k,d_0) + t\frac{q^k-1}{q-1}, k, d_0 + t.q^{k-1}]$ for given d_0, k, $t = t_0, t_0+1, t_0+2, \ldots$, and $[g_q(k+i, d_0.q^i + t.q^{k+i-1}), k+i, d_0.q^i + t.q^{k+i-1}]$ for given d_0, k, $t = t_0, t_0+1, t_0+2, \ldots$ and $i = i_e, i_e+1, \ldots$. This yields to the classification of all binary Griesmer codes with even weights up to dimension 6.

The paper is organized as follows: We give a preliminary information, basis definitions and theorems and some historical remarks in Section 1. In Section 2, we study the projective dual transform. We

describe some properties of codes and families of codes which are images under a dual transform. We develop a method for classification of Griesmer codes in Section 3. In the last section, we give some computational results.

1.1 Preliminaries

In this part, we give some basic definitions and theorems.

Definition 5. An automorphism of a linear code C is a sequence of the transformations from definition 3, which maps each codeword of C onto a codeword of C.

The set of all automorphisms of the code C forms a group which is called the automorphism group $Aut(C)$ of C.

The linear codes have simple description in the terms of generator matrices. For Griesmer codes, we can make the assumption that the codes don't have zero coordinates. This means that each column of a generator matrix of a code can be considered as a point in $PG(k-1,q)$. In geometrical aspect, we can define an $[n,k,d]_q$ code as a multiset C of n points in $PG(k-1,q)$ such that:

(a) each hyperplane of $PG(k-1,q)$ meets C in at most $n-d$ points;

(b) there is a hyperplane meeting C in exactly $n-d$ points.

This definition is equivalent to the previous one (see [15]). Two multisets of points from $PG(k-1,q)$, S and T, are said to be projectively equivalent if there exists a collineation $\pi \in P\Gamma L(k,q)$ which maps S onto T. Codes are equivalent if and only if the corresponding multisets of points are projectively equivalent.

The following Lemma holds:

Lemma 6. *Let G_1 and G_2 be generator matrices of the linear codes C_1 and C_2 without zero coordinates. C_1 and C_2 are equivalent iff*

$$G_2 = TG_1M\gamma,$$

where T is an invertible matrix, M is a monomial matrix and γ is a field automorphism.

Definition 7. *The number of the nonzero coordinates of a vector in $\mathbb{F}_q^n$ is called weight of the vector. Let A_i be the number of codewords of weight i in C. The ordered $(n+1)$-tuple of integers $(A_0, A_1, \ldots, A_n)$ is called the weight spectrum of C. Sometimes it is convenient to work with the so-called Hamming weight enumerator of C defined by*

$$W_C(x,y) = \sum_{i=0}^{n} A_i x^{n-i} y^i.$$

Definition 8. *Let C be an $[n,k,d]_q$-code over $\mathbb{F}_q$, G be a generator matrix of C and let $c \in C$ be a codeword of weight w. Then the residual code of C with respect to c, denoted by $Res(C;c)$, is the code generated by the restriction of G to the columns where c has a zero entry. If only the weight w of c is of importance, we will denote it by $Res(C;w)$.*

Lemma 9. *[12], [13] Suppose C is an $[n,k,d]_q$ code and suppose $\mathbf{c} \in C$ has weight w, where $d > w(q-1)/q$. Then $Res(C,\mathbf{c})$ is an $[n-w, k-1, d']_q$ code with $d' \geq d - w + \lceil w/q \rceil$.*

Definition 10. *A linear code is called projective if no two columns of its generator matrix are linearly dependent.*

1.2 Classification of optimal linear codes

In this section we consider the following problem: For a given set of parameters n, k, d and q find generator matrices of all nonequivalent q-ary $[n, k, d]$ codes. This problem has two main subproblems. First of them is to construct at least one code from any class of equivalence and the second one is the equivalence test.

There are theoretical results for classification of linear codes with very specific properties. Some of these results are related to perfect codes and MDS codes. The relation between finite projective geometries and projective codes is widely used for classification of projective Griesmer codes. In this direction we refer to a long series of papers by Hamada and coauthors which used the language of minihypers (see for example [17, 18]). Other geometric proofs can be found in [22, 24, 25]. For classification of some families of Griesmer codes we refer to [19]. Summarized classification results for binary linear codes can be found in [23] (see also [3]). There are some other sporadic results.

Practically, it is impossible to classify codes, which have many equivalent classes, without computer. There are several different types of algorithms which we use for classification. All these algorithms contain equivalence tests. We reduce the equivalence test of linear codes to isomorphism test of binary matrices (or bipartite graphs). In the general case, this is not trivial because it is related to the problem of finding weight spectra of the codes - a computationally intractable problem [2]. For the equivalence test of linear codes we use proper subsets of codewords. Such a subset has to be stable under the action of the code automorphism group and to generate the code as a vector space. In the binary case these subsets can be considered as a special kind of incidence matrices for bipartite graphs with m and n vertices (m is the number of the codewords in the subset). Practically, we reduce the q-ary case also to binary matrices ($\{0, 1\}$ matrices) and bipartite graphs.

The algorithms of the first type are so called *orderly* algorithms [27]. The specific thing here is that we can generate recursively a huge number of objects which have given inherited properties. The most important advantages of this approach are that we consider equivalence between small number of codes (only between children from the same parent), and it is very simple for parallelization.

The other type of algorithms is based on the fact that each linear code can be connected with hierarchy of subcodes or residual codes which are optimal or close to optimal. Our approach is a nontrivial back-track search closely related to dynamic programming. To restrict the search tree we cut some parts using equivalence up to extension. These algorithms have been implemented in the program **Q-extension** [6]. This program works well in many classification problems for reasonable time. **Q-extension** is an universal tool for solving many concrete different problems for classification.

But these algorithms can be used only for codes with relatively small parameters. To classify some Griesmer codes or even infinite families of Griesmer codes we combine them with a method which is based on properties of the projective dual transform.

1.3 Linear codes and projective dual transform

As mentioned earlier, any linear code can be considered as a multiset of points of the projective geometry $PG(k-1, q)$. There is a natural duality between points and hyperplanes in projective geometry: any point $a = (a_1, a_2, \ldots, a_k)$ defines hyperplane H_a which consists of all points $x = (x_1, x_2, \ldots, x_k)$ such that $(a, x) = \sum a_i x_i = 0$. This duality (known as projective duality or Delsarte duality) is useful to investigate projective two-weight codes (see [11], [22]). Brouwer and van Eupen [9] generalized Delsarte duality in the case of non-projective two-weight codes. Next generalization was done by Dodunekov and Simonis [15] (see also Nogin [28], Calderbank and Kantor [10]).

Projective dual transform (dual transform) has been used or studied in some other papers. See for example [4], [8], [20], etc. But still there are many open questions concerning its properties. For example, it is known that the dual transform is invertible but it is not clear what happens with families

of codes (see Section 2.4).

Dodunekov and Simonis [15] investigated some properties in the case when the transform has a smaller dimension than the original code. We also consider this case, but in a little different way. In contrast of [15], where they give a "matrix free presentation of the correspondence between full-length codes and projective multisets", we mainly use matrix presentation of $[n, k, d]_q$ linear codes, namely a $k \times n$ generator matrix or an $m \times n$ matrix which generates the code, where $m > k$.

2 Projective dual transform

In the beginning of this section we give some notations and definitions which are useful for the following propositions. Let us denote the set of all nonzero vectors in $\mathbb{F}_q^k$ with first nonzero coordinate 1 by $\widehat{S_k}$. We can consider these vectors as representatives of the points in the projective geometry $PG(k-1, q)$. Let S_k be the matrix whose columns are the lexicographically ordered elements of $\widehat{S_k}$. This matrix is a generator matrix of the simplex code with length $(q^k-1)/(q-1)$ and a unique nonzero weight q^{k-1}.

We say that the matrix G is in normal form if its columns are lexicographically ordered elements of $\widehat{S_k}$.

Remark 1. *In this research, we are interested in codes without zero coordinates. Any linear code of this type is equivalent to a code with generator matrix in normal form. We study properties of codes related to Hamming metric, that is why in some cases we consider codes up to equivalence. We define the code with parameters* $[0, 0, 0]$ *as a zero code with length zero. Length of the code* C *for us is the number of its nonzero coordinates (or we consider as length the effective length of the code).*

Let C be a linear $[n, k]_q$ code with a generator matrix G. Any vector $a \in \widehat{S_k}$ defines a codeword $c = aG \in C$. Let $\widehat{R_a}$ be the set of all solutions from $\widehat{S_k}$ of the equation $ax^{\mathrm{T}} = 0$. The elements of $\widehat{R_a}$, $|\widehat{R_a}| = \frac{q^{k-1}-1}{q-1}$, can be considered as representatives of points in projective subspace with projective dimension $k-2$, or a hyperplane. Similarly, we denote by R_a the matrix in normal form which have as columns the elements of $\widehat{R_a}$. If $\{r_1, r_2, \ldots, r_{(q^{k-1}-1)/(q-1)}\}$ are the vectors in $\widehat{R_a}$, then $\{r_1G, r_2G, \ldots, r_{(q^{k-1}-1)/(q-1)}G\}$ is the set of all nonproportional codewords of C which belong to a subcode C_a with dimension $k-1$.

The rows of the matrix $M_G = S_k^{\mathrm{T}}G$ are all nonzero nonproportional codewords of C. The matrix $M_{G_a} = R_a^{\mathrm{T}}G$ consists of these rows of M_G, which belong to the subcode C_a.

More generally, let G be an $m \times n$ matrix, $m \geq k$, which contains as rows k linearly independent codewords of a linear $[n, k \leq m]_q$ code C. Similarly, we can define the matrix $M_G = S_m^{\mathrm{T}}G$ and $M_{G_a} = R_a^{\mathrm{T}}G$ for $a \in \widehat{S_m}$.

If $k < m$ and $s = m - k$, the matrix $M_G = S_m^{\mathrm{T}}G$ will have

$$(q^s - 1)/(q - 1) \tag{1}$$

zero rows. The set R_G of the solutions of the homogeneous system $xG = 0$ forms a linear space with dimension $m - k = s$ since $rank(G) = k$, and $(q^s - 1)/(q - 1)$ of them are in the set $\widehat{S_m}$.

Let $aG = c$ for some $a \in \widehat{S_m}$. In this notation, for any vector $v \in a + R_G$ we have $vG = c$. Moreover, for any v there exists an element $\mu \in \mathbb{F}_q$, such that $\mu v \in \widehat{S_m}$ and $\mu vG = \mu c$. We can conclude that the matrix M_G contains exactly

$$q^s \tag{2}$$

rows proportional to c.

Now we will count the number of the nonzero entries in the matrix M_G. Let b be a column of G. The equation $xb = 0$ has q^{m-1} solutions in $\mathbb{F}_q^m$ and $(q^{m-1} - 1)/(q - 1)$ in $\widehat{S_m}$. It follows that the

number of nonzero elements in a column of M_G is $(q^m - q^{m-1})/(q-1) = q^{m-1}$. Hence the number of the nonzero entries in the matrix M_G is

$$nq^{m-1}. \tag{3}$$

Let us consider again the column of M_{G_a}, which correspond to the column b in G. Let $b \neq a$. The system

$$\left| \begin{array}{rcl} bx^{\mathrm{T}} & = & 0 \\ ax^{\mathrm{T}} & = & 0 \end{array} \right.$$

has q^{m-2} solutions in $\mathbb{F}_q^m$ and $(q^{m-2}-1)/(q-1)$ in $\widehat{S_m}$, hence the number of nonzero entries in the corresponding column in M_{G_a} is $(q^{m-1}-1)/(q-1)-(q^{m-2}-1)/(q-1) = (q^{m-1}-q^{m-2})/(q-1) = q^{m-2}$. If $b = a$, we obtain a zero column in M_{G_a}. Let n_a be the number of the nonzero columns in M_{G_a}. Then the number of the nonzero entries in M_{G_a} is:

$$n_a.q^{m-2} \tag{4}$$

If the vector a is linearly dependent with the columns of G, then the dimension of the subcode $C_a \subset C$ is $k-1$. Obviously, this is the only possibility when $m = k$. If $m > k$ and a is linearly independent with the columns of G, the code C_a coincides with C and therefore $n_a = n$.

Remark 2. *The effective length n_a of the code C_a is $n_a = n - \rho_a$, where ρ_a is the multiplicity of the vector a as a column of G. Actually, the number of nonzero entries in M_G and M_{G_a} depends only on the multiplicities of columns of G but not on the rank of G.*

2.1 Definition and correctness

Definition 11. *Let G be an $m \times n$ matrix with rank $k \leq m$, which generates a linear $[n,k,d]_q$ code C, and let the rows of the matrix $M_G = S_m^{\mathrm{T}} G$ have weights $w_1, w_2, \ldots, w_z$. Let $\alpha \neq 0$ and β be rational numbers such that $\alpha w_i + \beta$ is nonnegative integer for any $i = 1, \ldots, z$. We call dual transform the map $D_{\alpha,\beta,m}$, which transforms a linear code C to a linear code with a generator matrix in normal form, which consists of all elements $a \in \widehat{S_m}$, taken $\alpha wt(a.G) + \beta$ times. We say that the code $D_{\alpha,\beta,m}(C)$ is projective dual code of C. For given α, β, m (β - nonnegative integer), we transform the zero code $[0,0,0]$ to the code with a generator matrix S_m, taken β times.*

We will denote the length, dimension, minimum distance and generator matrix of the projective dual code with $D_{\alpha,\beta,m}(n)$, $D_{\alpha,\beta,m}(k)$, $D_{\alpha,\beta,m}(d)$ and $D_{\alpha,\beta,m}(G)$. It is clear that for $k = m$ we map the code C to a code with the same or smaller dimension. If $k < m$, we map the code C to a code with larger dimension. In that case, we can take the matrix G in the form $G = \binom{G'}{O}$ where G' is a generator matrix of C, and O is the $(m-k) \times n$ zero matrix.

The next theorem is very important for this work.

Theorem 12. *Let C_1 and C_2 be two equivalent q-ary codes without zero coordinates. Then their projective dual codes $D_{\alpha,\beta,m}(C_1)$ and $D_{\alpha,\beta,m}(C_2)$ for given α, β and m are also equivalent.*

Proof. Let G_1 and G_2 be $m \times n$ matrices which generate C_1 and C_2, respectively. Since the codes are equivalent, we have

$$G_2 = TG_1M\gamma,$$

where T is an invertible $m \times m$ matrix, M is an $n \times n$ monomial matrix and γ is a field automorphism. The multiplicity of the column $a \in \widehat{S_m}$ in the matrix $D_{\alpha,\beta,m}(G_2)$ coincides with the multiplicity of aT in the matrix $D_{\alpha,\beta,m}(G_1)$ (let's mention that the matrix $G_2 = TG_1M\gamma$ can be obtained from TG_1 with a permutation of the columns and a field automorphism). □

Notice that different generator matrices of the same code can lead to different but equivalent dual codes. That's why we will consider the dual codes up to equivalence. This theorem shows us that the dual transform does not depend on the matrix G but only on α, β and m. So we can take the matrix G to be in normal form.

Remark 3. *This transform is defined for projective codes in the case $k = m$ by Brouwer and van Eupen [9]. In the general case it has been studied by Dodunekov and Simonis [15].*

Remark 4. *The definition 11 is a little bit different from the previous known definitions for dual transform. It gives us the possibility to map a code with dimension k to a code with dimension m for $m > k$.*

2.2 Examples

Example 1. $[6, 3, 2]_2 \longrightarrow [24, 3, 12]_2 \quad \alpha = 1, \beta = 0, m = 3$

Let's consider the $[6, 3, 2]_2$ code C_1 with a generator matrix G_1 where

$$G_1 = \begin{pmatrix} 001111 \\ 110011 \\ 110101 \end{pmatrix}$$

Let fix $\alpha = 1$, $\beta = 0$, $m = 3$. For any weight w of the code, we have $\alpha w + \beta \geq 0$. In this case

$$S_3 = \begin{pmatrix} 0001111 \\ 0110011 \\ 1010101 \end{pmatrix}.$$

Let $v_1, v_2, \ldots, v_7$ be the columns of S_3. All nonzero codewords of C_1 can be presented as $v_i G_1$. The multiplicity of the vectors v_i as columns in the generator matrix of the dual code can be found as follows:

$v_1G_1 = (110101) \Rightarrow wt(v_1G_1) = 4 \Rightarrow \alpha.4 + \beta = 4$
$v_2G_1 = (110011) \Rightarrow wt(v_2G_1) = 4 \Rightarrow \alpha.4 + \beta = 4$
$v_3G_1 = (000110) \Rightarrow wt(v_3G_1) = 2 \Rightarrow \alpha.2 + \beta = 2$
$v_4G_1 = (001111) \Rightarrow wt(v_4G_1) = 4 \Rightarrow \alpha.4 + \beta = 4$
$v_5G_1 = (111010) \Rightarrow wt(v_5G_1) = 4 \Rightarrow \alpha.4 + \beta = 4$
$v_6G_1 = (111100) \Rightarrow wt(v_6G_1) = 4 \Rightarrow \alpha.4 + \beta = 4$
$v_7G_1 = (001001) \Rightarrow wt(v_7G_1) = 2 \Rightarrow \alpha.2 + \beta = 2$

So the projective dual code has the following generator matrix

$$D_{1,0,3}(C_1) = \begin{pmatrix} 000000000011111111111111 \\ 000011111100000000111111 \\ 111100001100001111000011 \end{pmatrix}$$

In the next example we show how to map a linear code to a code with larger dimension.

Example 2. $[6, 2, 4]_2 \longrightarrow [24, 3, 12]_2 \quad \alpha = 1, \beta = 0, m = 3$

Let's consider the binary $[6, 2, 4]$ code C_2 and the matrix G_2 which contains a basis of C_2, where

$$G_2 = \begin{pmatrix} 001111 \\ 110011 \\ 000000 \end{pmatrix}$$

In this case, as in the previous one, the weight of the rows of $M_G = S_3^{\mathrm{T}} G_2$ defines the multiplicity of any vector v_i as a column in the generator matrix of the dual code:

$v_1G_2 = (000000) \;\Rightarrow\; wt(v_1G_2) = 0 \;\Rightarrow\; \alpha.0 + \beta = 0$
$v_2G_2 = (110011) \;\Rightarrow\; wt(v_2G_2) = 4 \;\Rightarrow\; \alpha.4 + \beta = 4$
$v_3G_2 = (110011) \;\Rightarrow\; wt(v_3G_2) = 4 \;\Rightarrow\; \alpha.4 + \beta = 4$
$v_4G_2 = (001111) \;\Rightarrow\; wt(v_4G_2) = 4 \;\Rightarrow\; \alpha.4 + \beta = 4$
$v_5G_2 = (001111) \;\Rightarrow\; wt(v_5G_2) = 4 \;\Rightarrow\; \alpha.4 + \beta = 4$
$v_6G_2 = (111100) \;\Rightarrow\; wt(v_6G_2) = 4 \;\Rightarrow\; \alpha.4 + \beta = 4$
$v_7G_2 = (111100) \;\Rightarrow\; wt(v_7G_2) = 4 \;\Rightarrow\; \alpha.4 + \beta = 4$

The dual code has following generator matrix:

$$D_{1,0,3}(C_2) = \begin{pmatrix} 000000001111111111111111 \\ 111111110000000011111111 \\ 000011110000111100001111 \end{pmatrix}$$

2.3 Main properties of the projective dual codes

2.3.1 Length of the projective dual codes

Let C be an $[n, k, d]_q$ code and $D_{\alpha,\beta,m}(C)$ be its dual for some appropriate α, β and m. Then the length of the dual code is

$$D_{\alpha,\beta,m}(n) = \sum_{a \in \widehat{S_m}} (\alpha wt(aG) + \beta) = \alpha \sum_{a \in \widehat{S_m}} wt(aG) + \frac{q^m - 1}{q - 1}\beta = \alpha q^{m-1} n + \beta \frac{q^m - 1}{q - 1}, \tag{5}$$

since the sum $\sum_{a \in \widehat{S_m}} wt(a.G)$ coincides with the number of the nonzero entries of M_G, given with (3).

This formula shows that the length of the projective dual code of C does not depend on the weight spectrum but only on α, β, m and the length (effective) of the code. Actually, the length does not depend even on the dimension of the code.

2.3.2 Weights of the projective dual codes

Let G be an $m \times n$ matrix which generates the linear code C. Let $b_1, b_2, \ldots, b_{D_{\alpha,\beta,m}(n)}$ be the columns of the matrix $D_{\alpha,\beta,m}(G)$ and $a \in \widehat{S_m}$. Then $c = aD_{\alpha,\beta,m}(G)$ is a codeword in $D_{\alpha,\beta,m}(C)$ with weight $wt(c) = D_{\alpha,\beta,m}(n) - |\widehat{H}_a|$, where $\widehat{H}_a$ is the set of the columns b_i in the matrix $D_{\alpha,\beta,m}(G)$, such that $ab_i = 0$. The vector a defines also a set $\widehat{R_a} \subset \widehat{S_m}$ and a matrix M_{G_a}.

Using the formula (4), for cardinality of $\widehat{H}_a$ we obtain:

$$|\widehat{H}_a| = \sum_{b \in \widehat{R_a}} (\alpha wt(bG) + \beta) = \alpha \sum_{b \in \widehat{R_a}} wt(bG) + \frac{q^{m-1} - 1}{q - 1}\beta = \alpha q^{m-2} n_a + \beta \frac{q^{m-1} - 1}{q - 1}.$$

For the weights of the code we have:

$$wt(c) = \alpha q^{m-2}(qn - n_a) + \beta q^{m-1}. \tag{6}$$

The number of the nonzero columns n_a in $M_{G_a} = R_a^{\mathrm{T}} G$, $a \in \widehat{S_m}$, determines the number of the different weights of $D_{\alpha,\beta,m}(C)$ and the values of these weights. In another hand, n_a is equal to the length n of the code C minus the multiplicity of the vector a as a column of G. The number of the nonproportional codewords with the same weight wt in $D_{\alpha,\beta,m}(C)$ depends on the elements of $\widehat{S}_m$, which have the same multiplicity in G and this multiplicity corresponds to the weight wt.

The set $\widehat{H}_a$ consists of all columns b_i of the matrix $D_{\alpha,\beta,m}(G)$ which belong to $\widehat{R_a}$. Actually, $\widehat{H}_a$ consists of all elements $b \in \widehat{R_a}$, taken $\alpha wt(bG) + \beta$ times. Since the set R_a defines the subcode C_a of C, we can say that the subcode C_a corresponds to the residual code of $D_{\alpha,\beta,m}(C)$ with respect to the codeword $c = aD_{\alpha,\beta,m}(G)$. When $k = m$ and the dimension of $D_{\alpha,\beta,m}(C)$ is also k, the subcode $C_a \subset C$ has dimension $k-1$. Since any subcode of C with this dimension can be defined by a vector $a \in \widehat{S_k}$, the following theorem holds:

Theorem 13. *Let C be a linear $[n,k,d]_q$ code and $D_{\alpha,\beta,k}(C)$ be its projective dual code for given α, β and $k = m$. If the dimension of $D_{\alpha,\beta,k}(C)$ is also k, then there exists one to one correspondence between the subcodes of C with dimension $k-1$ and the residual codes of the code $D_{\alpha,\beta,k}(C)$.*

Proof. Let C_a be the subcode of C defined by $a \in \widehat{S_k}$. We have already mentioned that the dimension of C_a is $k-1$. We can correspond to C_a the residual code of $D_{\alpha,\beta,k}(C)$ with respect to the codeword $aD_{\alpha,\beta,k}(G)$. In our case, the different vectors $a \in \widehat{S_k}$ define different subcodes C_a and also different codewords $aD_{\alpha,\beta,k}(G)$ of $D_{\alpha,\beta,k}(C)$. The residual code for any codeword $a'D_{\alpha,\beta,k}(G)$ is the image of the subcode $C_{a'}$ of C with dimension $k-1$. □

We can continue this consideration for the subspaces of dimension $k-2$ which belong to the subspace of dimension $k-1$ and so on (see the example for dual codes of the subcodes for the binary Golay code in [20]).

Let's consider in details the case when C is a projective code. Then there are only two possibilities for the number of the nonzero columns of M_{G_a}: n - for any $a \in \widehat{S_m}$, which is not a column of G (their number is $\frac{q^m-1}{q-1} - n$) and $n-1$ - for all $a \in \widehat{S_m}$, which are columns of G (their number is n). For the weights in the dual code we obtain:

$$\{(\alpha n + \beta)q^{m-1} - \alpha n q^{m-2}, \quad (\alpha n + \beta)q^{m-1} - \alpha(n-1)q^{m-2}\}.$$

The exception is only the case of a simplex code or a repeated simplex code. Then its dual code is a constant weight or the zero code.

Lemma 14. *Let C be a linear $[n,k,d]_q$ and $D_{\alpha,\beta,m}(C)$ be its projective dual for given suitable α, β and m, $m > k$. Then the code $D_{\alpha,\beta,m-1}(C)$ is a residual code of $D_{\alpha,\beta,m}(C)$ with respect to a codeword c.*

Proof. We can choose (see Theorem 12) the $m \times n$ matrix G, which generates the code C in the following form: the first k rows form a basis of C and the other $m-k$ rows are zero vectors. Let $D_{\alpha,\beta,m}(C)$ and $D_{\alpha,\beta,m-1}(C)$ be two projective dual codes of C. To construct the second code, we use the matrix G' which is obtained from G by deleting the last row. Then the code $D_{\alpha,\beta,m-1}(C)$ is a residual of $D_{\alpha,\beta,m}(G)$ with respect to the codeword $(000\ldots001)D_{\alpha,\beta,m}(G)$.

The set $\widehat{R_a}$ for $a = (000\ldots001)$ consists of all vectors from $\widehat{S_m}$ which have 0 in the last position. So the set $\widehat{H_a}$, defining the residual code have only vectors with 0 in the last coordinate. Actually, they can be considered as vectors from S_{m-1} with additional position 0. □

2.3.3 Dimension of the projective dual codes

It is clear, that the dimension of the projective dual code $D_{\alpha,\beta,m}(k)$ of the code C is at most m and it is exactly m when $M_{D_{\alpha,\beta,m}(G)} = S_m^{\mathrm{T}} D_{\alpha,\beta,m}(G)$ does not have zero rows.

Suppose that the dimension of $D_{\alpha,\beta,m}(k)$ is smaller than m. Then there exists a vector $a \in \widehat{S_m}$ such that the corresponding codeword in $D_{\alpha,\beta,m}(C)$ has weight 0, i.e.

$$\alpha q^{m-2}(q.n - n_a) + \beta q^{m-1} = 0 \;\Leftrightarrow\; \rho_a = n - n_a = -\frac{\beta}{\alpha}q - (q-1)n$$

Let $s = m - D_{\alpha,\beta,m}(k)$. The number of zero rows in $M_{D_{\alpha,\beta,m}(G)}$ is given with the formula (1) and the number of repeated nonzero rows is given by (2). The number of the elements in $\widehat{S_m}$, which have multiplicity $-\frac{q\beta}{\alpha} - (q-1)n$ as columns in G is $(q^s-1)/(q-1)$ and they can be considered as representatives of points in projective geometry with projective dimension $s-1$. The number of the elements in $\widehat{S_m}$, which have different multiplicity in G will be q^s. In this research, we will not consider the case when $m > k$ and the dimension of the dual codes is smaller than m. When $k = m$, the following lemma holds:

Lemma 15. *Let C be a linear $[n,k,d]_q$ code, $\alpha \neq 0$ and β be rational numbers, such that the dimension of the dual code $D_{\alpha,\beta,k}(k) < k$, and $s = k - D_{\alpha,\beta,k}(k)$. Then the elements of $\widehat{S_k}$, which have multiplicity $-\frac{q\beta}{\alpha} - (q-1)n$ as columns of the matrix G, are representatives of all point in the projective geometry $PG(s-1,q)$. If T is the set of these elements and $a \in \widehat{S_k}$, $a \notin T$, has the multiplicity ρ_a as a column of G, then any element $c = \nu(a + \mu.b) \in \widehat{S_k}$, where $b \in T, \nu, \mu \in \mathbb{F}_q$, has the same multiplicity ρ_a.*

Proof. If $b \in T$, then

$$wt(bD_{\alpha,\beta,k}(G)) = \alpha q^{k-2}(q.n - n_b) + \beta q^{k-1} = \alpha q^{k-2}(qn - n - \frac{q\beta}{\alpha} - (q-1)n) + \beta q^{k-1} = 0$$

Hence for $a \in \widehat{S_k}, a \notin T$, we have

$$\nu(a + \mu b)D_{\alpha,\beta,k}(G) = \nu a D_{\alpha,\beta,k}(G).$$

But the weights of these two vectors depend on the multiplicities of $\nu(a + \mu b)D_{\alpha,\beta,k}(G)$ and νa as columns in G, therefore these multiplicities are equal. □

2.4 Invertibility

It is known that the dual transform is invertible [15]. A projective dual code $C' = D_{\alpha,\beta,m}(C)$ of the $[29,5,20]_5$ code C via dual transform defined by $\alpha = -1/5$, $\beta = 5$ and $m = 4$, is a linear code with parameters $[257,4,205]_5$. In the inverse transform we have $D_{-1/5,41,4}(C') = C$. But not all $[257,4,205]_5$ codes goes to $[29,5,20]_5$ codes via this mapping. In this section, we consider in more details the invertibility of the dual transform.

Let the $m \times n$ matrix G generates an $[n, k \leq m, d]_q$ code C and let $D_{\alpha,\beta,m}(C)$ be its projective dual code with generator matrix $D_{\alpha,\beta,m}(G)$ for given α, β and m. For each codeword $c = aD_{\alpha,\beta,m}(G)$ of weight $wt(c) = \alpha q^{m-2}(q.n - (n - \rho_a)) + \beta q^{m-1}$, there are ρ_a copies of the column a in G. If the matrix G contains ρ_a copies of the column a, it defines the codeword $c = aD_{\alpha,\beta,m}(G)$ of weight $wt(c) = \alpha q^{m-2}(q.n - (n - \rho_a)) + \beta q^{m-1}$. The codeword $c = aD_{\alpha,\beta,m}(G)$ by the inverse transform, which is defined by $\alpha' = \frac{q^{2-k}}{\alpha}$ and $\beta' = -\frac{q\beta}{\alpha} - (q-1)n$, determines ρ_a copies of the column a in G.

Theorem 16. *For given $\alpha \neq 0, \beta, m > 0$ and $n > 0$, let Γ_i be the family of all codes with dimension k, $i \leq k \leq m$, effective length n, and nonzero weights $w_1, w_2, \ldots, w_z$ such that $\alpha w_l + \beta$ is a nonnegative integer for $l = 1,2,\ldots,z$. Let Θ_j be the family of the codes with length $\alpha q^{m-1}n + \beta\frac{q^m-1}{q-1}$, dimension k, $j \leq k \leq m$, and nonzero weights $\alpha q^{m-2}(q.n - n_a) + \beta q^{m-1}$, $0 \leq n_a \leq n$. Then there exists one to one correspondence between Γ_i and Θ_j, where:*

- *$i = 1$ when β is a nonnegative integer, and $i = m$ when β is negative or not integer.*
- *$j = 1$ when $\beta' = -\frac{q\beta}{\alpha} - (q-1)n$ is a nonnegative integer, and $j = m$ when $-\frac{q\beta}{\alpha} - (q-1)n$ is negative or not integer.*

Proof. Let C be a linear code with length n and dimension $k \leq m$. Its dual $D_{\alpha,\beta,m}(C)$ is a code with length $n' = \alpha q^{m-1} n + \beta \frac{q^m-1}{q-1}$ and weights $wt(c) = \alpha q^{m-2}(q.n - n_a) + \beta q^{m-1}$.

Let us consider the dual transform $D_{\alpha',\beta',m}$, defined by α' and β', where $\alpha' = \frac{q^{2-m}}{\alpha}$ and $\beta' = -\frac{q\beta}{\alpha} - (q-1)n$. This transform maps the codes with length $n' = \alpha q^{m-1} n + \beta \frac{q^m-1}{q-1}$ and weights $\alpha q^{m-2}(q.n - n_a) + \beta q^{m-1}, 0 \leq n_a \leq n$, to the codes with length n. It can be verified by direct calculations that the weights w of the given type are the only weights for which $\alpha' w + \beta'$ are nonnegative integers.

Let's now consider the dimensions of the images. We have the following cases:

a) $\beta \geq 0$, $\beta \in \mathbb{Z}$, and $\beta' \geq 0$, $\beta' \in \mathbb{Z}$. Then the dual transform $D_{\alpha,\beta,m}$ can be applied to all codes in Γ_1, and the images of the codes, which have columns in their generator matrix with multiplicity β', will have smaller dimension. Similarly, we can apply the inverse dual transform to all codes in Θ_1.

b) $\beta \geq 0$, $\beta \in \mathbb{Z}$, and $\beta' < 0$ or β' is not integer. The dual transform $D_{\alpha,\beta,m}$ can be applied to all codes in Γ_1, but their images have dimension m, since no column has multiplicity β' in the generator matrix of a code in this family. Moreover, the inverse transform $D_{\alpha',\beta',m}$ can be applied only to codes with dimension m. Hence $D_{\alpha,\beta,m}$ gives a bijection between the families Γ_1 and Θ_m.

c) $\beta < 0$ or β is not integer, and $\beta' \geq 0$, $\beta' \in \mathbb{Z}$. This case is similar to the previous one.

d) $\beta < 0$ or β is not integer, $\beta' < 0$ or β' is not integer. Then the transforms $D_{\alpha,\beta,m}$ and $D_{\alpha',\beta',m}$ can be applied only to codes with dimension m and therefore in this case we consider the families Γ_m and Θ_m. □

Let us fix rational numbers α and β and a positive integer m. Let Δ be the family of all codes of length n, dimension $k \leq m$ when β is a nonnegative integer, and dimension m when β is a negative integer or not integer, such that $\alpha w + \beta$ is a nonnegative integer for any nonzero weight w of a codeword. Then α, β and m define also the family Λ of the codes dual to the codes from Δ.

Let α_1 and β_1 (β_1 is nonnegative iff β is nonnegative) be rational numbers such that $\alpha_1 w + \beta_1$ is a nonnegative integer for all nonzero weights of the code C of length n and dimension as given above, only when $C \in \Delta$. Then α_1, β_1 and m define the family Λ_1 of codes which are dual to all codes in Δ.

Corollary 17. *There is one-to-one correspondence between Λ and Λ_1.*

This corollary leads to families of codes with specific properties. For example, any code in one of the families is a t-replicated code from the other family.

This corollary is quite useful for classifications of Griesmer codes whose weights are divisible by a given integer.

Let C be a linear code with a generator matrix G, such that its dual for given α and β has the same dimension. In this case the matrix $D_{\alpha,\beta,k}(G)$ is uniquely determined. In opposite, if we know $D_{\alpha,\beta,k}(G)$, then G is uniquely determined. Let us consider the matrix $U = D_{\alpha,\beta,k}(G)^{\mathrm{T}} G$. Then $U^{\mathrm{T}} = G^{\mathrm{T}} D_{\alpha,\beta,k}(G)$. The matrix U generates the code C, and U^{T} generates $D_{\alpha,\beta,k}(C)$. The inverse transform in this case can be considered as a transposition of the matrix U.

Let C be a projective code and let $g_1, g_2, \ldots, g_n$ be the columns of G. These columns define all subcodes of C with dimension $k-1$ and effective length $n-1$ (g_i defines the subcode of all codewords with zero in the i-th coordinate). Then $D_{\alpha,\beta,k}(C)$ is a two-weight code and the defined g_i determine all codewords of $D_{\alpha,\beta,k}(C)$ with one of the weights and $h \in \widehat{S_k} \setminus \{g_1, g_2, \ldots, g_n\}$ determine the codewords of $D_{\alpha,\beta,k}(C)$ with the other weight.

Dodunekov and Simonis introdused a new notion of self-duality in coding theory [15].

Definition 18. *The linear code C is projective self-dual if it is equivalent to its projective dual code for some α, β and $m = k$.*

Definition 19. *The linear code C is formally projective self-dual if the parameters and weight spectrum of C and its dual $D_{\alpha,\beta,k}(C)$ are the same for some $\alpha, \beta, m = k$.*

The next lemma gives necessary and sufficient conditions for the formally self-duality.

Proposition 20. *Suppose C is not a replicated simplex code. Then C is a formally projective self-dual code for some α and β $(m = k)$ if and only if it is formally projective self-dual for*

$$\alpha = \epsilon q^{1-\frac{k}{2}}, \ \beta = -\frac{q-1}{1+q^{k-1}\alpha}n + \epsilon\lfloor\frac{n}{(q^k-1)/(q-1)}\rfloor, \tag{7}$$

where $\epsilon = +1$ or -1.

Proof. It is a trivial generalization of Proposition 6 in [15]. □

The following assertions for the formally projective self-dual codes hold:

- If C is a formally projective self-dual code, then the code C' obtained by a juxtaposition of C with t copies of the simplex code of dimension k is also formally projective self-dual.
- Let the projective code C be formally projective self-dual. Then C is a two-weight or a simplex code.
- Let the projective two-weight code C be formally projective self-dual and D be its projective dual code. Then C and D have isomorphic automorphism groups [7].

As so many parameters of the formally projective self-dual code C and its dual code D coincide, the following natural question arises: Is any formally projective self-dual code C projective self-dual? The answer is negative. Let us consider the $[68, 8, 32]_2$ codes C_1 and C_2 with weight spectrums $W_1 = W_2 = 1 + 187z^{32} + 68z^{40}$, and generator matrices:

$$G_1 = \begin{pmatrix}
00000000000000000000000000001111111111111111111111111111111111111111 \\
00000000111111111111111111110000000000000000000011111111111111111111 \\
00111111000000111111111111110000000000111111111100000000001111111111 \\
01000111011111000000111111100000111110001111110000011111000111111 \\
11111111000011000111100000110111100011011000111100111001111110000111 \\
11011011011101011011100111110001101111100001001101011000010110111001 \\
11101101001111111100101011111010110011101010010001001010100010011111 \\
01110100110100101101110101011011110101110000110101011011001110101011
\end{pmatrix}$$

$$G_2 = \begin{pmatrix}
00000000000000000000000000000000000011111111111111111111111111111111 \\
00000000000000000000111111111111111100000000000000001111111111111111 \\
00000000000011111111000000001111111100000000111111110000000011111111 \\
00000011111000001110000011100011111000111110000011100000111000111111 \\
00000110001100011011000110110110001100100111001110010011100100100111 \\
00011010000100101101001011010010110101001011000111100100101111100001 \\
01100000110100111011001111011011011001000101010010100011001001001010 \\
10101010010001011101010011000100001100010011101100001101110011110010
\end{pmatrix}$$

It is easy to see that they are projective dual (for $\alpha = 1/8, \beta = -4$ and $m = 8$), but not equivalent.

Only one example for formally projective self-dual code with more than 2 nonzero weights has been known until now. Using a computer search, we construct many three-weight codes of this type. A formally projective self-dual code with more than three nonzero weights is not known.

3 Griesmer codes

Let C_1 and C_2 be linear codes of the same dimension k, and G_1 and G_2 be their generator matrices. The linear code C is obtained by juxtaposition of these two codes if $(G_1|G_2)$ is its generator matrix. Obviously, if C_2 is the simplex code, then all codes obtained by juxtaposition of C_1 and C_2, using different generator matrices of both codes, are equivalent.

Lemma 21. *[12] Let C be a Griesmer $[g_q(k,d),k,d]_q$ code and C_1 be the code obtained by juxtaposition of C with the t replicated simplex code of dimension k. Then C_1 is a $[g_q(k,d)+t\frac{q^k-1}{q-1},k,d+tq^{k-1}]_q$ Griesmer code.*

Proof.

$$g_q(k,d+tq^{k-1})=\sum_{i=0}^{k-1}\lceil\frac{d+tq^{k-1}}{q^i}\rceil=\sum_{i=0}^{k-1}(\lceil\frac{d}{q^i}\rceil+tq^{k-1-i})$$

$$=\sum_{i=0}^{k-1}\lceil\frac{d}{q^i}\rceil+t\frac{q^k-1}{q-1}=g_q(k,d)+t\frac{q^k-1}{q-1}$$

□

Lemma 22 (Lemma 1.3 [12]). *Let $d\le(t+1)q^{k-1}$. Any generator matrix of the $[s+g_q(k,d),k,d]_q$ code C does not contain more than $t+1+s$ proportional columns.*

These lemmas give us a motivation to consider Griesmer codes in intervals for minimum distance and length, respectively:

$$\{1,2,\dots,q^{k-1}\},\{q^{k-1}+1,q^{k-1}+2,\dots,2q^{k-1}\},\dots,\{(t-1)q^{k-1}+1,(t-1)q^{k-1}+2,\dots,tq^{k-1}\},\dots$$

$$\{g_q(k,1),\dots,\frac{q^k-1}{q-1}\},\{g_q(k,q^{k-1}+1),\dots,2\frac{q^k-1}{q-1}\},\dots,\{g_q(k,((t-1)q^{k-1}+1),\dots,t\frac{q^k-1}{q-1}\},\dots$$

Any of these intervals ends with the t replicated simplex code.

It is easy to see that a Griesmer code of length $n=(t+1)\frac{q^k-1}{q-1}-r\le(t+1)\frac{q^k-1}{q-1}=g_q(k,(t+1)q^{k-1})$ and minimum distance d, $tq^{k-1}\le d\le(t+1)q^{k-1}$, does not have more than $t+1$ proportional columns in its generator matrix and a Griesmer code in the first interval ($t=0$) is projective. Such a Griesmer code can be presented as the $t+1$ replicated simplex code without r coordinates. What happens when $r=t$? In this case, the code contains a whole copy of the simplex code. This means that all Griesmer codes with length $(t+1)\frac{q^k-1}{q-1}-r$ can be obtained from the Griesmer codes with length $t\frac{q^{k-1}-1}{q-1}-r$ and a copy of the simplex code.

Let $d_0\le q^{k-1}$. We will consider the following parameters of codes $[g_q(k,d_0)+t\frac{q^k-1}{q-1},k,d_0+t.q^{k-1}]_q$ for $t=1,2,\dots$. It is known that for any k there is a constant t_0 such that for any $t\ge t_0$ a Griesmer code of this type exists. We prove a theorem about the number of the codes.

Theorem 23. *For given k and $d_0<q^{k-1}$, there exists a constant L such that for any $t\ge L$ the number of the nonequivalent $[g_q(k,d_0)+t\frac{q^k-1}{q-1},k,d_0+t.q^{k-1}]_q$ Griesmer codes is a constant.*

Proof. Let $d_0 = q^{k-1} - d'$ and $g_q(k, d_0) = \frac{q^k-1}{q-1} - n'$. Then $n' = \frac{q^k-1}{q-1} - g_q(k, d_0)$. For the parameters of the codes we obtain $[n = (t+1)\frac{q^k-1}{q-1} - n', k, d = d_0 + t.q^{k-1}]_q$.

Let us choose $L = n' - 1$. The length of the corresponding Griesmer code for $t = L + 1$ will be $n_t = (n'+1)\frac{q^k-1}{q-1} - n'$. To obtain this code, we delete n' columns from $n'+1$ copies of the simplex code. This means that we have at least a copy of a simplex code in the Griesmer code. Hence the number of the nonequivalent Griesmer codes with dimension k and minimum distance $d_0 + (L+1).q^{k-1}$ is the same as the number of the Griesmer codes in the same dimension and minimum distance $d_0 + L.q^{k-1}$. □

Theorem 24. *For given k, there is a constant L_k such that for any n the number of the nonequivalent Griesmer codes of length n and dimension k is not more than L_k.*

Proof. Let $l_1, l_2, \ldots, l_{q^{k-1}-1}$ be the number of the nonequivalent codes with parameters $[(t+1)\frac{q^k-1}{q-1} - (t+1), k, d_0 + t.q^{k-1}]_q$ for $d_0 = 1, 2, \ldots, q^{k-1} - 1$, respectively. The theorem is true for $L_k = max\{l_1, l_2, \ldots, l_{q^{k-1}-1}\}$. □

3.1 Classification of Griesmer codes

It is clear that the classification of families of Griesmer codes of type $[g_q(k, d_0)+t\frac{q^k-1}{q-1}, k, d_0+t.q^{k-1}]_q$ for $1 \leq d_0 \leq q^{k-1}$ and $t < L$, leads to the classification of all Griesmer codes in a given dimension k. But the problem for classification of codes with length larger than the length of the simplex code is a quite difficult even for the codes over small fields. We show that the classification of Griesmer codes with minimum distance $d_0 + t.q^{k-1}$ is equivalent to the problem for classification of codes with fixed length n', dimension not larger than k and weights divisible by ξ, where ξ is a power of the characteristic of the field. In many cases the second problem is easier than the first one. The length n' is different for different value of d_0. That is why the classification of $[g_q(k+i, d_0.q^i + t.q^{k+i-1}), k+i, d_0.q^i + t.q^{k+i-1}]_q$ codes becomes possible for the small values of n'. The computational cost depends also on the values of the constant L and this constant is quite important for us.

An integer is called a *divisor* of a code C if it divides all weights of C. We use the following remarkable theorem.

Theorem 25. *Let C be an $[n, k, d]$ code over $\mathbb{F}_p$, p a prime, meeting the Griesmer bound. If $p^e \mid d$, then p^e is a divisor of C.*

This theorem has been proved for binary case by Dodunekov and Manev [14] and for nonbinary case by H. Ward [33] (see also [24]).

For codes over $\mathbb{F}_q$ when $q = p^e$, the following theorem holds.

Theorem 26. *Let C be an $[n, k, d]_q$ Griesmer code over $GF(q)$, $q = p^e$, where p is prime. Let $p^s \mid d$, for $p^s \geq q$. Then $\frac{p^{s+1}}{q}$ is a divisor of C.*

This theorem has been proved for $q = 4$ by H. Ward in 2001 [34], and for any power of a prime $q = p^e$ by Xiaoyu Liu in 2006 [26].

As a corollary, we have that if an integer $l = p^i$ is a divisor of a projective Griesmer code with minimum distance d_0, then l is a divisor of all Griesmer codes with minimum distance $d_0 + tq^{k-1}$, $t \geq 1$.

Let C be a projective $[n = g_q(k, d_0), k, d_0]$ Griesmer code over $\mathbb{F}_q$, $q = p^e$ (p - prime) with nonzero weights $\{d_0, w_2, \ldots, w_z\}$, and $\delta = p^s$ be a divisor of C, where $0 \leq s \leq k-1$. Choose $\alpha = \frac{1}{\delta}$, $\beta = -\frac{d_0}{\delta} = d_1$. For the length of the dual code we have

$$n' = \frac{n}{\delta}q^{k-1} - \frac{d_0}{\delta}\frac{q^k - 1}{q - 1},$$

and the weights of $D_{\alpha,\beta,k}(C)$ are

$$w_\rho = \frac{1}{\delta}q^{k-2}(q.n-(n-\rho)) - \frac{d_0}{\delta}q^{k-1}.$$

Since C is projective, $\rho = 0$ or 1. It follows that $D_{\alpha,\beta,k}(C)$ is a two-weight code with nonzero weights

$$w_0 = \frac{1}{\delta}q^{k-2}(q.n-(n-1)) - \frac{d_0}{\delta}q^{k-1} = \frac{q^{k-2}}{\delta}(n(q-1)+1) - \frac{q^{k-1}d_0}{\delta}$$

$$w_1 = \frac{1}{\delta}q^{k-2}(q.n-n) - \frac{d_0}{\delta}q^{k-1} = \frac{q^{k-2}}{\delta}n(q-1) - \frac{q^{k-1}}{\delta}d_0 = w_0 - \frac{q^{k-2}}{\delta}$$

It is clear that $\xi = \frac{q^{k-2}}{\delta}$ is a divisor of $D_{\alpha,\beta,k}(C)$. We will prove now that the codes with length n', weights divisible by ξ and dimension not greater than k determine all Griesmer codes with minimum distance $d_0 + tq^{k-1}$.

If $s = e(k-1)$, C is a projective Griesmer code iff C is equivalent to the simplex code with parameters $[\frac{q^k-1}{q-1}, k, q^{k-1}]_q$. We do not consider this case because the dual code of C is the zero code. So let $s \leq e(k-2)$.

The inverse transform is defined by $\alpha' = \frac{q^{2-k}}{\alpha} = \delta q^{2-k} = p^{s+e(2-k)}$ and $\beta' = -\frac{q\beta}{\alpha} - n(q-1) = \frac{q\delta d_0}{\delta} - n(q-1) = d_0 q - nq + n$. We have to mention that β' does not depend on δ.

Let C' be a linear code with length $n' = D_{\alpha,\beta,k}(n)$, dimension k, and nonzero weights divisible by $\xi = \frac{q^{k-2}}{\delta}$, and the largest of them is $w_0 = \frac{q^{k-2}}{\delta}(n(q-1)+1) - \frac{q^{k-1}d_0}{\delta}$.

For the maximal number of the different weights in C' we have

$$\frac{w_0}{\xi} = (\frac{q^{k-2}}{\delta}(n(q-1)+1) - \frac{q^{k-1}d_0}{\delta})\frac{\delta}{q^{k-2}} =$$

$$-(d_0 q - nq + n) + 1 = 1 - \beta'.$$

We can consider the nonzero weights of C' in the form $w_0 - i\xi = (1-\beta')\xi - i\xi = (1-\beta'-i)\xi$ for $0 \leq i \leq -\beta'$, or $(-\beta'-i)\xi$ for $0 \leq i < -\beta'$.

Suppose that the minimum distance d of C' is smaller than w_1 or $d = (-\beta'-u)\xi$ for $0 < u < -\beta'$. Let $\alpha = 1/\xi = \alpha'$ and $\beta = \beta' + r \geq -(d)/(\xi) = \beta' + u$ and β integer. For all weights w of C' $\alpha w + \beta$ is a nonnegative integer. The projective dual code $D_{\alpha',\beta'+r,k}(C')$ has length $n + r\frac{q^k-1}{q-1}$.

For the weights of the dual codes we have the following formula:

$$\begin{aligned} wt(c) &= \alpha' q^{k-2}(n'(q-1)+\rho) + (\beta'+r)q^{k-1} \\ &= \delta q^{2-k}q^{k-2}((\tfrac{q^{k-1}}{\delta}n - \delta d_0\tfrac{q^k-1}{q-1})(q-1)+\rho) + (d_0 q - nq + n + r)q^{k-1} \\ &= \delta(\tfrac{q^{k-1}}{\delta}n(q-1) - \delta d_0(q^k-1)+\rho) + d_0 q^k - nq^k + nq^{k-1} + rq^{k-1} \\ &= q^{k-1}n(q-1) - d_0(q^k-1) + \rho\delta + d_0 q^k - nq^{k-1}(q-1) + rq^{k-1} \\ &= d_0 + \rho\delta + rq^{k-1} \end{aligned}$$

All parameters in the last sum are positive and the weight zero is impossible. So the dual code has the same dimension. The parameter ρ shows the multiplicity of any coordinate and the minimum value of ρ, denoted by ρ_{min}, shows the number of the copies of the simplex code in C'. If $\rho > 0$, we have parameters for the code which contradict the Griesmer bound. Hence the dual code $D_{\alpha',\beta'+r,k}(C')$ has parameters $[n + r\frac{q^k-1}{q-1}, k, d_0 + rq^{k-1}]_q$.

If we replace r by $r+1$, the multiplicity of any column in the dual code will be the previous multiplicity plus 1, so the dual code $D_{\alpha',\beta'+r+1,k}$ can be obtained by juxtaposition of $D_{\alpha',\beta'+r,k}$ and the $[\frac{q^k-1}{q-1}, k, q^{k-1}]_q$ simplex code.

We have already mentioned in Theorems 25 and 26 that there is a connection between the divisors of the minimum distance and the divisors of a Griesmer code. Let us consider codes over a field with $q = p^e$ elements, $e \geq 1$, where p is a prime. If $\frac{q}{p}\delta$ is a divisor of the minimum distance of a Griesmer code C, where $\delta = p^s$, $s \geq 0$, then δ is a divisor of all weights of C. Let's consider the family of all Griesmer codes of dimension k and minimum distance d_0, where d_0 is divisible by $\frac{q}{p}\delta$. We denote by Ω_t the family of linear codes with dimension k, length

$$n' = \frac{n}{\delta}q^{k-1} - \frac{d_0(q^k-1)}{\delta(q-1)},$$

and nonzero weights $\{w_0, w_1, \ldots, w_t\}$, where $n = g_q(k, d_0)$, $w_0 = \frac{n}{\delta}q^{k-2}(q-1) - \frac{d_0}{\delta}q^{k-1}$, $w_i = w_0 - i\frac{q^{k-2}}{\delta}, i = 1, \ldots, t$, $1 \leq t \leq -\beta'$, and $\beta' = d_0 q - n(q-1)$. For $t = 1-\beta'$, we denote by Ω_t the family of all linear codes with length n', nonzero weights $w_0, w_1, \ldots, w_{t-1}$, and dimension $\leq k$. It is clear that:

$$\Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega_{-\beta'} \subseteq \Omega_{1-\beta'}$$

Let Ψ_t be the family of all Griesmer codes with dimension k and minimum distance $d_0 + tq^{k-1}$, $t \geq 0$. The next theorem gives the connections between families Ψ and Ω_t.

Theorem 27. *There is one-to-one correspondence between the families Ω_1 and Ψ_0, Ω_2 and Ψ_1, ..., $\Omega_{1-\beta'}$ and $\Psi_{-\beta'}$, respectively. Moreover, all codes in the family $\Psi_{-\beta'+j}$, $j \geq 0$, can be obtained from the codes of $\Psi_{-\beta'}$ by juxtaposition with j copies of the simplex code with dimension k.*

Proof. Let us consider $\alpha = \frac{1}{\delta}$, $\beta = -\frac{d_0}{\delta}$, $n = g_q(k, d_0)$ and $m = k$. The family Γ_k in the notation of Theorem 16 consists of all codes with dimension k, length $n = g_q(k, d)$, and nonzero weights $w_1 < w_2 < \cdots < w_z$ such that $\alpha w_l + \beta$ is nonnegative integer for $l = 1, 2, \ldots, z$. This means that these codes have minimum distance $w_1 = d_0$, and they belong to the family Ψ_0. In other words, the family Θ_k coincides with Ω_1. Following theorem 16, there is one-to-one correspondence between these two families. We have to notice that $-\frac{q\beta}{\alpha} - (q-1)n = qd_0 - (q-1)n < 0$.

The existence of one-to-one correspondence between the families Ω_{t+1} and Ψ_t for $t = 0, 1, \ldots, -\beta'-1$ follows similarly from Theorem 16, where $\alpha = \frac{1}{\delta}$, $\beta_t = -\frac{d_0}{\delta} - t\frac{q^{k-1}}{\delta}$, $n = g_q(k, d_0 + tq^{k-1})$ and $m = k$.

For $t \geq -\beta'$, we have $-\frac{q\beta_t}{\alpha} - (q-1)n \geq 0$, and therefore there is one-to-one correspondence between the families Γ_k and Θ_1, which coincide with $\Psi_{-\beta'+j}$ and $\Omega_{1-\beta'}$, respectively. □

In the following example, we discuss the classification of the families Ψ_t of Griesmer codes with parameters $[11 + 31t, 3, 8 + 25t]_5$ for $t = 0, 1, 2, \ldots$.

Example 3. *The projective codes in $\{\Psi_0, \Psi_1, \ldots\}$ are the codes from the first family, namely the Griesmer codes with parameters $[11, 3, 8]_5$. So we have $k = 3$, $d_0 = 8$, $\alpha = 1$, $\delta = 1$, $\beta = -8$ and $m = 3$. The dual of a $[11, 3, 8]_5$ code is $D_{1,-8,3}(C) = [27, 3, 20]_5$ with weights divisible by $\xi = q^{k-2}/\delta = 5$.*

To classify all codes in Ψ_t, we need the classification of all codes with length 27, and with weights divisible by $\xi = q^{k-2}/\delta = 5$. So we obtain the following results:
Ω_1 consists of the $[27, 3, 20]_5$ codes with weights from the set $\{0, 20, 25\}$. The number of nonequivalent codes with these parameters is 2. Shortly we will write $\Omega_1 = \{[27, 3, \{20, 25\}] - 2\}$
$\Omega_2 = \{[27, 3, \{20, 25\}] - 2, [27, 3, \{15, 20, 25\}] - 14\}$,
$\Omega_3 = \{[27, 3, \{20, 25\}] - 2, [27, 3, \{15, 20, 25\}] - 14\}], [27, 3, \{10, 15, 20, 25\}]25\}$
$\Omega_4 = \{[27, 3, \{20, 25\}] - 2, [27, 3, \{15, 20, 25\}] - 14, [27, 3, \{10, 15, 20, 25\}] - 25, [27, 3, \{5, 10, 15, 20, 25\}] - 9\}$
$\Omega_5 = \{[27, 3, \{20, 25\}] - 2, [27, 3, \{15, 20, 25\}] - 14, [27, 3, \{10, 15, 20, 25\}] - 25, [27, 3, \{5, 10, 15, 20, 25\}] -$

$9, [27, 2, \{20, 25\}] - 1, [27, 2, \{15, 20, 25\}] - 1, [27, 2, \{10, 15, 20, 25\}] - 1, [27, 2, \{5, 10, 15, 20, 25\}] - 0\}$

These families correspond to the following families of Griesmer codes. After the parameters we write the number of the nonequivalent codes.
Ψ_0 - $\{[11, 3, 8] - 2\}$
Ψ_1 - $\{[42, 3, 33] - 16\}$
Ψ_2 - $\{[73, 3, 58] - 41\}$
Ψ_3 - $\{[104, 3, 83] - 50\}$
Ψ_4 - $\{[135, 3, 108] - 53\}$
Ψ_5 - $\{[166, 3, 133] - 53\}$
...

Let $r = 1 - \beta'$. More generally, we can consider the families Ω_1^i, Ω_2^i, ..., Ω_{r-1}^i, Ω_r^i, which consist of codes with the same length and weights as the codes in Ω_1, Ω_2, ..., Ω_r, respectively, and dimension $k + i$ ($\leq k + i$ for the last family), and also Ψ_t^i as a family of all Griesmer codes with dimension $k + i$ and minimum distance $d_0 q^i + t q^{k+i-1}$, $t = 0, 1, 2, 3, \ldots$ and $i = 0, 1, 2, \ldots$. Then we have the following theorem:

Theorem 28. *There is one-to-one correspondence between the families Ω_1^i and Ψ_0^i, Ω_2^i and Ψ_1^i, ..., Ω_r^i and Ψ_{r-1}^i. Moreover, all codes in the family Ψ_{r-1+j}^i can be obtained from the codes of Ψ_{r+1}^i by juxtaposition with j copies of the simplex code with dimension $k + i$. If $\Omega_1^i, \Omega_2^i, \ldots, \Omega_{r-1}^i$ are empty for $i \geq i_e$ then all families Ψ_t^i have the same number of codes (up to equivalence) for $t = r - 1, r, r + 1, r + 2, \ldots$ and $i = i_e, i_e + 1, i_e + 2, \ldots$.*

Proof. Let C_i be a Griesmer code with dimension $k + i$ and minimum distance $d_0 q^i$, and $\frac{q}{p}\delta$ be a divisor of d_0, $\delta = p^s$. Using Theorem 16, we have that $\delta_i = \delta q^i$ is a divisor of all weights of C_i. For given $\alpha = \frac{1}{\delta_i}$ and $\beta = -\frac{d_0 q^i}{\delta_i}$, the projective dual code $D_{\alpha,\beta,m}(C_i)$ has length

$$n_i' = \frac{n}{\delta_i} q^{k+i-1} - \frac{d_0 q^i}{\delta_i} \frac{q^{k+i} - 1}{q - 1} = \frac{n}{\delta . q^i} q^{k-1} q^i - \frac{d_0 q^i}{\delta . q^i} \frac{q^{k+i} - 1}{q - 1} =$$

$$= \frac{q^{k-1}}{\delta} g_q(k + i, d_0 q^i) - \frac{d_0}{\delta} \frac{q^{k+i} - 1}{q - 1}$$

Let C_{i+1} be the Griesmer code with dimension $k + i + 1$ and minimum distance $d_0 . q^{i+1}$. Then $\delta_{i+1} = \delta q^{i+1}$ is a divisor of C_{i+1}. For given $\alpha = \frac{1}{\delta_{i+1}}$ and $\beta = -\frac{d_0 . q^{i+1}}{\delta_{i+1}}$, its projective dual code has length

$$n_{i+1}' = \frac{n}{\delta_{i+1}} q^{k+i+1-1} - \frac{d_0 q^{i+1}}{\delta_{i+1}} \frac{q^{k+i+1} - 1}{q - 1} = \frac{n}{\delta . q^{i+1}} q^{k-1} q^{i+1} - \frac{d_0 q^{i+1}}{\delta . q^{i+1}} \frac{q^{k+i+1} - 1}{q - 1} =$$

$$= \frac{q^{k-1}}{\delta} g(k + i + 1, d_0 q^{i+1}) - \frac{d_0}{\delta} \frac{q^{k+i+1} - 1}{q - 1}$$

$$= \frac{q^{k-1}}{\delta} (d_0 q^{i+1} + \lceil \frac{d_0 q^{i+1}}{q} \rceil + \cdots + \lceil \frac{d_0 q^{i+1}}{q^{k+i}} \rceil) - \frac{d_0}{\delta} \frac{q^{k+i+1} - 1}{q - 1}$$

$$= \frac{q^{k-1}}{\delta} (d_0 q^{i+1} + d_0 q^i + \cdots + \lceil \frac{d_0 q^i}{q^{k+i-1}} \rceil) - \frac{1}{\delta} \frac{d_0 q^{k+i+1} - d_0}{q - 1}$$

$$= \frac{q^{k-1}}{\delta} (g(k + i, d_0 q^i) + d_0 q^{i+1}) - \frac{1}{\delta} \frac{d_0 q^{k+i+1} - d_0}{q - 1}$$

$$= \frac{q^{k-1}}{\delta} g(k+i, d_0 q^i) - \frac{d_0}{\delta} \frac{q^{k+i}-1}{q-1} = n'$$

We can conclude that the projective dual codes of all Griesmer codes with minimum distance $d_0 q^i$ and dimension $k+i$ for $\alpha = \frac{1}{\delta q^i}$, $\beta = -\frac{d_0 . q^i}{\delta q^i}$ and $m = k+i$ have the same length equal to the length of the dual of the Griesmer code with minimum distance d_0 and dimension k. We can obtain the same for the weights of the dual codes. Hence the weights of these codes have a divisor $\xi = \frac{q^{k-2}}{\delta}$.

For any positive i we can apply Theorem 27 for the families Ω_1^i, Ω_2^i, ..., Ω_{r-1}^i, Ω_r^i and the families of Griesmer codes Ψ_t^i , for $t = 0, 1, 2, 3, \ldots$. If $\Omega_1^i, \Omega_2^i, \ldots, \Omega_{r-1}^i$ are empty, then $|\Omega_r^i| = |\Omega_r^{i-1}|$. The number of codes in the last family is the same as the number of Griesmer codes with dimension $k+i-1$ and minimum distance $d_0 q^{i-1} + t q^{k+i-2}$. Hence the families Ψ_t^i and Ψ_t^{i-1} have the same number of equivalent classes and the number of all codes in Ψ_t^i are the same for $i \geq i_e$. □

All codes with minimum distance $d_0 q^i$ for $i = 1, 2, \ldots$ can be obtained from two-weight codes with length n' and weights divisible by $\xi = \frac{q^{k-2}}{\delta}$, with dual transform defined by $\alpha' = \frac{q^{2-k}}{\alpha} = \delta q^{2-k} = p^{s+e(2-k)}$, $\beta' = -\frac{q\beta}{\alpha} - n(q-1) = d_0 q - g_q(k, d_0)(q-1)$ and $m = k+i$. Let B be the code with dimension k, length n' and nonzero weights ξr and $\xi(r+1)$ $(r = -\beta' + 1)$. Let us consider its dual codes B_1 for $m = k+i$, and B_2 for $m = k+i-1$, $i > 1$. According to Lemma 14, B_2 will be residual of B_1 with respect to a codeword with minimum weight.

Let B_s be a subcode of B with dimension $k-1$ and effective length n'. A subcode with these parameters exists if B has a column with multiplicity one. Its dual is a residual of the code C with respect to a codeword with minimum distance and also reaches the Griesmer bound. We have the following Lemma:

Lemma 29. *If C is a Griesmer code, its residual code with respect to a codeword with minimum distance is also a Griesmer code.*

Proof. It is based on the fact that $\lceil \frac{a}{cb} \rceil = \lceil \frac{\lceil \frac{a}{c} \rceil}{b} \rceil$ □

In the next example we discuss the classification of the following family of Griesmer codes:

$$\{\psi_t^i = \{[g_5(4+i, 70.5^i + t.5^{i+3}), 4+i, 70.5^i + t.5^{i+3}]_5\},\ t = 0, 1, 2, \ldots, i = 1, 2, \ldots\}. \tag{8}$$

Example 4. Let $k = 4$ and $d_0 = 70$. Consider the projective Griesmer codes over $\mathbb{F}_5$ with parameters $[88, 4, 70]_5$. Take $\alpha = 1/\delta = 1/5, \beta = -14$. The dual $D_{1/5,-14,4}(C)$ of an $[88, 4, 70]_5$ code C is a $[16, 4, 10]_5$ code with weights divisible by $\xi = 5^{k-2}/\delta = 5$.

To classify all codes in this family, we need the classification of all codes over $\mathbb{F}_5$ with length 16, and weights divisible by 5. We obtain the following results:

Ω_1 consists of $[16, 4, 10]_5$ codes with weights from the set $\{0, 10, 15\}$. But such codes do not exist. Shortly, we will write $\Omega_1 = \{[16, 4, \{10, 15\}] - 0\}$. This means that Griesmer $[88, 4, 70]_5$ codes do not exist. Nevertheless, these parameters define a family of Griesmer codes, which is not empty. For the other sets we have:

$\Omega_2 = \{[16, 4, \{5, 10, 15\}] - 1\}$, $\Omega_3 = \{[16, \leq 4, \{5, 10, 15\}] - 6\}$.

$\Omega_1^1 = \{[16, 5, \{10, 15\}] - 0\}$, $\Omega_2^1 = \{[16, 5, \{5, 10, 15\}] - 0\}$, $\Omega_3^1 = \{[16, \leq 5, \{5, 10, 15\}] - 6\}$. More precisely, $\Omega_3^1 - \{[16, 5, \{10, 15\}] - 0$, $[16, 5, \{5, 10, 15\}] - 0$, $[16, 4, \{10, 15\}] - 0$, $[16, 4, \{5, 10, 15\}] - 1$, $[16, 3, \{10, 15\}] - 0$, $[16, 3, \{5, 10, 15\}] - 3$, $[16, 2, \{10, 15\}] - 1$, $[16, 2, \{5, 10, 15\}] - 1\}$.

As we see, the sets Ω_1^1 and Ω_2^1 are empty, therefore $|\Omega_3^1| = |\Omega_3|$.

This family corresponds to the family of Griesmer codes (8). After the parameters we write the number of the nonequivalent codes. We also list the weight spectrums of the codes in Ψ_2^3 and Ψ_2^4.

$\Psi_0^1 = \varnothing$
$\Psi_1^1 = \varnothing$
$\Psi_2^1 = \{[2000, 5, 1600] - 6\}$
$\Psi_3^1 = \{[2781, 5, 2225] - 6\}$
$\Psi_2^3 = \{[50000, 7, 40000] - 6\}$
$1 + 78100z^{40000} + 16z^{40625} + 8z^{43750}$
$1 + 78092z^{40000} + 24z^{40625} + 8z^{43125}$
$1 + 78096z^{40000} + 20z^{40625} + 4z^{43125} + 4z^{43750}$
$1 + 78092z^{40000} + 24z^{40625} + 8z^{43125}$
$1 + 78096z^{40000} + 24z^{40625} + 4z^{46250}$
$1 + 78100z^{40000} + 20z^{40625} + 4z^{46875}$
$\Psi_2^4 = \{[250000, 8, 200000] - 6\}$
$1 + 390600z^{200000} + 16z^{203125} + 8z^{218750}$
$1 + 390592z^{200000} + 24z^{203125} + 8z^{215625}$
$1 + 390596z^{200000} + 20z^{203125} + 4z^{215625} + 4z^{218750}$
$1 + 390592z^{200000} + 24z^{203125} + 8z^{215625}$
$1 + 390596z^{200000} + 24z^{203125} + 4z^{231250}$
$1 + 390600z^{200000} + 20z^{203125} + 4z^{234375}$
...

3.2 Evaluation of the constant L by dual transform

Let us consider again the family of Griesmer codes with minimum distance $d_0 + tq^{k-1}$, $t = 1, 2, \ldots$. We have already mentioned that there is a constant L such that for any $t \geq L$ the number of the nonequivalent Griesmer codes is a constant. Now we will evaluate more precisely this constant and study its properties. We begin with the following Theorem:

Theorem 30. *The constant L depends only on k and d_0, and $L = L(k, d_0) = g_q(k, d_0)(q-1) - d_0 q$.*

Proof. Theorem 27 shows that we can take $L \leq g_q(k, d_0)(q-1) - d_0 q$. The existence of a Griesmer code with minimum distance $d_0 + (g_q(k, d_0)(q-1) - d_0 q)q^{k-1}$ which does not contain the simplex code as a punctured code is enough to prove the theorem. Let C be a Griesmer code with minimum distance $d = d_0 + tq^{k-1}$, $t \leq g_q(k, d_0)(q-1) - d_0 q$. Using the dual transform defined by $\alpha = 1$, $\beta = -d$, and $m = k$, we have two cases:

1) $D_{\alpha,\beta,k}(k) = k$. Let G be a generator matrix of C. If $wt(aG) = d$, where $a \in \widehat{S}_k$, we take the vector a as a column in $D_{\alpha,\beta,k}(G)$ with multiplicity 0. As we have already noticed, then the vector a defines a subcode of $D_{\alpha,\beta,k}(C)$ with dimension $k-1$ and effective length $n' = D_{\alpha,\beta,k}(n)$. Using the dual transform $D_{q^{2-k},0,k}$, we map this subcode into a Griesmer code with dimension k and minimum distance $d_0 + (g_q(k, d_0)(q-1) - d_0 q)q^{k-1}$ which does not contain the simplex code as a punctured code. For the last claim we use similar arguments as above, namely: If the matrix H with k rows generates the subcode, its rows are linearly dependent and therefore $aH = 0$ for a nonzero $a \in \widehat{S}_k$. Then the multiplicity of a in the generator matrix of the dual code will be 0, that's why this vector is not a column in that matrix.

2) $D_{\alpha,\beta,k}(k) < k$. This is possible only if $t = g_q(k, d_0)(q-1) - d_0 q$. We can use similar approach for $D_{\alpha,\beta,k}(C)$ as for the subcode in the previous case. □

The value of the constant L depends on the dimension and the minimum distance, and also on the divisors of d_0. We can expect that the value of L will be larger for smaller values of d_0.

Proposition 31. *For $d_0 < q^{k-2}$ (or equivalently $d_0 + (q-1)q^{k-2} < q^{k-1}$) we have $L(k-1, d_0) = L(k, d_0 + (q-1)q^{k-2})$.*

Proof.

$$L(k, d_0 + (q-1)q^{k-2}) = g_q(k, d_0 + (q-1)q^{k-2})(q-1) - (d_0 + (q-1)q^{k-2})q =$$

$$= (q-1)\sum_{i=0}^{k-1} \lceil \frac{d_0 + (q-1)q^{k-2}}{q^i} \rceil - (d_0 + (q-1)q^{k-2})q =$$

$$= (q-1)\sum_{i=0}^{k-2} (\lceil \frac{d_0}{q^i} \rceil + (q-1)q^{k-2-i}) + \lceil \frac{d_0 + (q-1)q^{k-2}}{q^{k-1}} \rceil - (d_0 + (q-1)q^{k-2})q =$$

$$= g_q(k-1, d_0)(q-1) + (q-1)(q^{k-1} - 1) + q - 1 - d_0 q - (q-1)q^{k-1} =$$

$$= g_q(k-1, d_0)(q-1) - d_0 q = L(k-1, d_0)$$

□

The Proposition above shows that the constant L is the same for the family of Griesmer codes defined by $d_{0_1} = q^{k-2} - d_0$ with dimension $k-1$ and $d_{0_2} = q^{k-1} - d_0$ with dimension k. The values of L are the same for any family of Griesmer codes with dimension j defined by the minimum distance $q^{j-1} - h$ for any h such that $h < q^{j-2}$.

Now we study the parameters of the dual code of a projective Griesmer code which will help us to prove some additional properties for the constant L.

Proposition 32. *Let C_1 and C_2 be projective Griesmer codes with dimension k and minimum distance d_0 and $d_0 + q^{k-2}$, $d_0 \leq q^{k-2}$, respectively. Denote by n', w_1', w_2' the length and the nonzero weights of $D_{1,-d_0,k}(C_1)$, and by n'', w_1'', w_2'' the length and the nonzero weights of $D_{1,-d_0-q^{k-2},k}(C_2)$. Then*

$$n' - n'' = w_1' - w_1'' = w_2' - w_2'' = q^{k-2}.$$

Proof. The length n' of the code $D_{1,-d_0,k}(C_1)$ is

$$n' = q^{k-1} g_q(k, d_0) - d_0 \frac{q^k - 1}{q - 1}$$

For the length n'' of the dual code $D_{1,-d_0-q^{k-2},k}(C_1)$ we have

$$n'' = q^{k-1} g_q(k, d_0 + q^{k-2}) - (d_0 + q^{k-2}) \frac{q^k - 1}{q - 1}$$

$$= q^{k-1} \sum_{i=0}^{k-1} \lceil \frac{d_0 + q^{k-2}}{q^i} \rceil - (d_0 + q^{k-2}) \frac{q^k - 1}{q - 1}$$

$$= q^{k-1} \sum_{i=0}^{k-2} (\lceil \frac{d_0}{q^i} \rceil + q^{k-2-i}) + q^{k-1} \lceil \frac{d_0 + q^{k-2}}{q^{k-1}} \rceil - (d_0 + q^{k-2}) \frac{q^k - 1}{q - 1}$$

$$= q^{k-1} \sum_{i=0}^{k-1} \lceil \frac{d_0}{q^i} \rceil - q^{k-1} \lceil \frac{d_0}{q^{k-1}} \rceil + q^{k-1} \lceil \frac{d_0 + q^{k-2}}{q^{k-1}} \rceil + q^{k-1} \frac{q^{k-1} - 1}{q - 1} - d_0 \frac{q^k - 1}{q - 1} - q^{k-2} \frac{q^k - 1}{q - 1}$$

$$= q^{k-1} g_q(k, d_0) + q^{k-1} (\lceil \frac{d_0 + q^{k-2}}{q^{k-1}} \rceil - \lceil \frac{d_0}{q^{k-1}} \rceil) - d_0 \frac{q^k - 1}{q - 1} - q^{k-2}$$

$$= q^{k-1} g_q(k, d_0) + q^{k-1}(1 - 1) - d_0 \frac{q^k - 1}{q - 1} - q^{k-2} = n' - q^{k-2}.$$

Similarly, we prove the formula for the weights. □

Corollary 33. *If $d_0 < q^{k-1}$ then $L(k+1, d_0) - L(k, d_0) = q - 1$.*

The next claim gives the connection between the family of Griesmer codes with dimension k, defined by d_0, and the family with dimension $k+1$, defined by $d_0 q$.

Proposition 34. *Let $d_0 < q^{k-1}$. Then $L(k, d_0) = L(k+1, d_0 q)$.*

Proof. $L(k+1, d_0 q) = g_q(k+1, d_0 q)(q-1) - d_0 q^2 =$
$(d_0 q + \lceil \frac{d_0 q}{q} \rceil + \cdots + \lceil \frac{d_0 q}{q^k} \rceil)(q-1) - d_0 q^2 = d_0 q(q-1) + g_q(k, d_0)(q-1) - d_0 q^2 = L(k, d_0)$. □

3.3 Structure of the dual of a Griesmer code

In this section, we present some facts which are useful for construction and classification of the dual of Griesmer codes.

Proposition 35. *Let C be a Griesmer code with dimension k and minimum distance $d_0 + iq^{k-1}$, $i \geq 0$. Then C has at least one coordinate with multiplicity $i+1$.*

Proof. The length $n = g_q(d_0 + iq^{k-1})$ of a Griesmer code with minimum distance $d_0 + iq^{k-1}$ is larger than $i(q^k - 1)/(q-1)$. Hence there are coordinates with multiplicity larger then i. Theorem 22 shows that their multiplicities are exactly $i+1$. □

Let Υ is the family of the Griesmer codes with dimension k and minimum distance $d_0 + tq^{k-1}$, $0 < d_0 < q^{k-1}$, $t \geq 0$. From the previous section we know that all codes from Υ can be obtained from the codes with the same length n and weights divisible by ξ. Proposition above shows that each of these codes contains a codeword with the largest weight w_0.

Let C be the Griesmer code with minimum distance qd. The punctured codes with $1, 2 \ldots q-1$ coordinates with minimum distances $qd-1, dq-2, \ldots dq-(q-1)$, respectively, also achieve the Griesmer bound. For the code C we can find all nonequivalent punctured with one coordinate codes. This number is equal to the number of orbits of C under the action of the automorphism group of C. The next theorem says how we can obtain the dual of the punctured code of C using the projective dual of the code C.

Theorem 36. *Let C be a linear code with minimum distance d, and $D_{1,-d+1,k}(C_1)$ be a projective dual of the code C_1 which is a puncturing of C with one coordinate. Then the generator matrix of $D_{1,-d+1,k}(C_1)$ can be considered in two parts such that the first one is a generator matrix of $D_{1,-d,k}(C)$ and the second one generates the simplex code with dimension $k-1$.*

Proof. Let G be a generator matrix of C and a be the column of G which is punctured. The vectors $x \in \widehat{S_k}$ such that $xa^{\mathrm{T}} = 0$, generate a subspace of $\mathbb{F}_q^k$ with dimension $k-1$ and so their number is $(q^{k-1} - 1)/(q-1)$. Hence the matrix H whose columns are the vectors x, generates the q-ary simplex code of dimension $k-1$.

If $x \in \widehat{S_k}$ then the codeword $xG \in C$ has 0 in the position of the column a if and only if $xa^{\mathrm{T}} = 0$. Therefore when $xa^{\mathrm{T}} = 0$, we take this vector in the generator matrix of $D_{1,-d,k}(C)$ with multiplicity $wt(xG) - d$ and its multiplicity in the generator matrix of $D_{1,-d+1,k}(C_1)$ will be $wt(xG) - d + 1$. When $xa^{\mathrm{T}} \neq 0$, then both multiplicities are equal to $wt(xG) - d$. Hence the generator matrix of $D_{1,-d+1,k}(C_1)$ is $(D_{1,-d,k}(G) \mid H)$. □

With direct calculations we obtain:

Proposition 37. *Let C_1 and C_2 be projective Griesmer codes with dimension k and minimum distances d_1 and $d_2 = d_1 - 1$, respectively, $d_1 \not\equiv 1 \pmod q$. Denote by n', w'_1, w'_2 the length and the nonzero weights of $D_{1,-d_1,k}(C_1)$ and by n'', w''_1, w''_2 the length and the nonzero weights of $D_{1,-d_2,k}(C_2)$. Then*

$$n' - n'' = \frac{q^{k-1}-1}{q-1}, \quad w'_1 - w''_1 = w'_2 - w''_2 = q^{k-2}.$$

In the binary case, because of the parity check bit, we can obtain the dual of a Griesmer code with even minimum distance d from the dual of a Griesmer code with odd minimum distance $d-1$. In the non binary case in this way we can obtain only projective duals of extendable codes (for extendibility see for example [25]).

Remark 5. *The MacDonald codes [31] are projective Griesmer codes with minimum distance $q^{k-1}-q^u$, $1 \le u \le k-2$. For given k, we can define generalized MacDonald codes as Griesmer codes with minimum distance $tq^{k-1} - q^u$. For this type of codes we have that if $d_0 = q^{k-1} - q^u$ then $L = 0$, $1 \le u \le k-2$. The generalized MacDonald codes form the family:*

$$\{[g_q(k+i, (q^{k-1}-1)q^i + tq^{k+i-1}),\ k+i,\ (q^{k-1}-1)q^i + tq^{k+i-1}]_q,\ k = 2,3,\dots,\ t,i = 0,1,2,\dots\}.$$

3.4 Weight spectrums of the codes in the considered families

The multiplicities of the coordinates completely determine the spectrum of the dual code for given α, β and m. That's why the *spectrum of multiplicities* is used in this paper. We do not write number of coordinates with multiplicity 0 - their number is the number of all coordinates minus the number of the other coordinates in the generator matrix.

The linear code with a generator matrix

$$G = \begin{pmatrix} 111100 \\ 110011 \\ 101011 \end{pmatrix}$$

has spectrum of multiplicities $S = 4^1 1^2$. This means that G has 4 columns with multiplicity one and 1 column with multiplicity two. To find the weight spectrum of the dual code for given α and β, we need the parameters of the codes, the divisor of the weights ξ and the spectrum of multiplicities. We write this information in the following way $[6,3,2]_2, \xi = 2 - 4^1 1^2$.

Let C be an $[n,k,d]_q$ code and let its projective dual code $D_{\alpha,\beta,m}(C)$, $m \ge k$, have nonzero weights $w_1 < w_2 < \dots < w_t$. Then the weight spectrum of the projective dual code $D_{\alpha,\beta,m+1}(C)$ depends on $A_{w_1}, A_{w_2}, \dots, A_{w_t}$, and more precisely $A_{w'_1} = A_{w_1} + \frac{q^{m+1}-1}{q-1} - \frac{q^m-1}{q-1}$, $A_{w'_2} = A_{w_2}, \dots, A_{w'_t} = A_{w_t}$, where $w'_1 < w'_2 < \dots < w'_t$ are the nonzero weights in $D_{\alpha,\beta,m+1}(C)$.

4 Calculational results and tables

We classify all binary Griesmer codes up to dimension 5 using our program $Q-Extension$. Moreover, we classify many different Griesmer codes and families of Griesmer codes over fields with 2, 3 and 4 elements.

In Table 2 we present the parameters of the dual codes of all projective Griesmer codes over $\mathbb{F}_2$ with dimension $2 \le k \le 6$ for $\alpha = 1$, $\beta = -d$, $m = k$. The duals of MacDonald codes have only one nonzero weight and their parameters are written in the form $[n,k,0;w]$ where w is the only nonzero weight. In the other cases the dual codes are two weight codes.

As it is proved above in the paper, to classify all Griesmer codes from the family $\{[g_2(k, d + tq^{k-1}), k, d+tq^{k-1}], t = 0,1,\dots\}$, we need to classify all codes with effective length $n' = D_{1,-d,k}(g_2(k,d))$,

dimension $k' \leq k$, and weights divisible by $w_2 - w_1$. The number of the nonequivalent codes with these properties for $k = 5$ are listed in Table 3 (see column 3 in Table 2). Moreover, we classify the codes with the same length but for dimension 6. These codes are dual to even Griesmer codes via the dual transform $D_{1/2,-d,6}$.

In Appendix, we present parameters of families of Griesmer codes. In the binary case these are the families with dimension $k = 6 + i$, $i = 0$ or $i \geq 0$, and even minimum distance. Moreover, we give the values of α and β which define the used dual transform. The codes with the parameters listed below determine all codes from the corresponding families. Using Theorem 28 and the number of codes from Table 3, we can obtain the number of the Griesmer codes with dimension $k \leq 5$.

Example 5. *For the* $[26, 5, 8]_2$ *code* C*, we have the following information:*
$[26, 5, 8]_2$ $\xi = 8$*;* $10^2 1^6$*;* $7^2 3^4$*; TOTAL – 2*

This information shows us that there exist exactly two codes with effective length 26, dimension 5 and weights divisible by 8 with different spectrums of multiplicities: $10^2 1^6$, $7^2 3^4$. It follows from table 2 that these codes are projective dual to Griesmer codes from the family, defined by the parameters $[g_2(5, 10), 5, 10]_2$. Fixing $\alpha = 1/8$, $\beta = -1$, $m = 5$, we can find the parameters and weight spectrums of the Griesmer codes with minimum distance 10 ($t = 0$) and dimension 5. For the lengths we have $n' = (1/8).2^4.26+(-1)(2^5-1) = 21 = g_2(5, 10)$, and for the weights $(1/8).8(2.26-26+\rho)-16 = \rho+10$, where ρ is a multiplicity of a column in the dual code. The dual of the code with spectrum of multiplicities $(20^0)10^2 1^6$ is a Griesmer code with weight spectrum $A_0 = 1$, $A_{10} = 20$, $A_{12} = 10$, $A_{16} = 1$.

References

[1] L. D. Baumert, R. J. McEliece, A note on the Griesmer bound, IEEE Trans. Inform. Theory 19 (1973) 134–135.

[2] E. R. Berlekamp, R. J. McEliece and H. C. van Tilborg, On the inherent intractability of certain coding problems, IEEE Trans. Inform. Theory 24 (1978) 384–386.

[3] A. Betten, The classification of small linear codes, `http://www.math.colostate.edu/~betten/`.

[4] I. Boukliev, S. Dodunekov, T. Helleseth, O. Ytrehus, On the [162,8,80] codes, IEEE Trans. Inform. Theory 43 (1997) 2055–2057.

[5] I. Bouyukliev and D. Jaffe, Optimal binary linear codes of dimension at most seven, Discrete Mathematics 226 (2001) 51–70.

[6] I. Bouyukliev and J. Simonis, Some new results for optimal ternary linear codes, IEEE Trans. Inform. Theory 48 (2002) 981–985.

[7] I. Bouyukliev, V. Fack, W. Willems, J. Winne, Projective two-weight codes with small parameters and their corresponding graphs, Designs, Codes and Cryptography 41 (2006) 59–78.

[8] I. Bouyukliev and J. Simonis, Some new results on optimal codes over $\mathbb{F}_5$, *Designs, Codes and Cryptography* 30 (2003) 97–111.

[9] A.E. Brouwer and M. van Eupen, The correspondence between projective codes and 2-weight codes, *Designs, Codes and Cryptography* 11 (1997) 261–266.

[10] A. R. Calderbank, W. M. Kantor, The geometry of two-weight codes, Bull. London Math. Soc. 18 (1986) 97–122.

[11] P. Delsarte, Weights of linear codes and strongly regular normed spaces, *Discrete Math.* 3 (1972) 47–64.

[12] S. M. Dodunekov, Optimal Linear Codes, Doctor of Math. Sciences Thesis, Institute of Mathematics, Sofia, Bulgaria, 1985.

[13] S. Dodunekov, Minimal block length of a linear q-ary code with specified dimension and code distance, *Problems Inform. Transmission* 20 (1984) 239–249.

[14] S. Dodunekov, N. Manev, Minimum possible block length for some distance, Problems of Information Transmission 20 (1984) 8–14.

[15] S. Dodunekov and J. Simonis, Codes and projective multisets, *Electron. J. Combin.* 5(1) (1998) (electronic).

[16] J.H. Griesmer, A bound for error-correcting codes, *IBM J. Res. Develop.* 4 (1960) 532–542.

[17] N. Hamada, A survey of recent work on characterization of minihypers in PG(t; q) and nonbinary linear codes meeting Griesmer bound, *J. Combin. Inform. Syst. Sci.* 18 (1993) 161–191.

[18] N. Hamada, T. Helleseth, Ø. Ytrehus, Characterization of $\{2(q+1)+2, 2; t, q\}$minihypers in $PG(t; q)$ $(t \leq 3, q \in \{3, 4\})$, Discrete Math. 115 (1993) 175–185.

[19] T. Helleseth, Further classifications of codes meeting the Griesmer bound, *IEEE Trans. Inform. Theory* 30 (1984) 395–403.

[20] R. Hill and E. Kolev, A survey of recent results on optimal linear codes, Combinatorial Designs and their Applications, Milton Keynes, 1997, Chapman & Hall/CRC Res. Notes Math. 403, Chapman & Hall/CRC, Boca Raton (1999) 127–152.

[21] R. Hill and P. Lizak, Extensions of linear codes, Proc. IEEE International Symp. Inf. Theory, Whistler, BC (1995) pp. 345.

[22] R. Hill, Caps and codes, *Discrete Math.*, 22 (1978) 111–137.

[23] D. B. Jaffe, Binary linear codes: new results on nonexistence, Draft version accessible through the author's web page `http://www.math.unl.edu/djaffe`, April 14,1997 (Version 0.4) Dept. of Math. and Statistics, University of Nebraska, Lincoln.

[24] I. Landjev. The geometric approach to linear codes. J. Geom., to appear.

[25] I. Landjev, Linear codes and multisets of points in finite geometres, Doctor of Math. Sciences Thesis, Institute of Mathematics, Sofia, Bulgaria, 2000.

[26] X. Liu, Divisibility of Griesmer Codes, preprint, 2006.

[27] B. D. McKay, Isomorph-free exhaustive generation, *J. Algorithms* 26 (1998) 306–324.

[28] D. Nogin, Weight/multiplicity duality, Proceedings of the Sixth International Workshop, Algebraic and Combinatorial Coding Theory, Pskov, Russia, September 1998, 195–198.

[29] N.J.A. Sloane, Is there a (72,36), $d = 16$ self-dual code? *IEEE Trans. Info. Theory* 19 (1973) 251.

[30] G. Solomon and J. J. Stiffler, Algebraically punctured cyclic codes, *Inform. and Control* 8 (1965) 170–179.

[31] F. Tamari, On linear codes which attain the Solomon-Stifler bound, Discrete Math. 49 (1984) 179–191.

[32] R. R. Varshamov, Problems of the general theory of linear coding, Extended abstract of a PhD Thesis, Moscow State University,(1959).

[33] H. N. Ward, Divisibility of codes meeting the Griesmer bound, *J. Comb. Theory Ser. A* 83 (1998) 79–93.

[34] H. N. Ward, Divisible codes - a survey, *Serdica Math. J.* 27 (2001) 263–278.

Appendix

$\Psi_t = \{[7+63t, 6, 2+32t]_2\}$ $|\Psi_t| = 1021$ for $t \geq 3$; $\alpha = 1/2, \beta = -(1+16t)$;

$[49,4,8]_2$ $\xi = 8$; $8^1 4^3 2^7 1^{15}$; $8^1 3^3 3^7 1^{11}$;TOTAL-2
$[49,4,16]_2$ $\xi = 8$; $7^1 4^3 3^7 1^9$; $6^1 5^3 2^5 1^7 1^{11}$; $6^1 4^3 2^5 3^7$; $6^1 6^3 1^5 1^7 1^{13}$; $5^1 6^3 2^5 1^7 1^9$; $6^1 6^3 2^5 1^{15}$;TOTAL-6
$[49,4,24]_2$ $\xi = 8$; $4^1 6^3 4^5 1^7$;TOTAL-1
$[49,5,8]_2$ $\xi = 8$;TOTAL-65
$[49,5,16]_2$ $\xi = 8$;TOTAL-69
$[49,5,24]_2$ $\xi = 8$; $10^1 10^2 5^3 1^4$;TOTAL-1
$[49,6,8]_2$ $\xi = 8$;TOTAL-603
$[49,6,16]_2$ $\xi = 8$;TOTAL-273
$[49,6,24]_2$ $\xi = 8$;$35^1 7^2$;TOTAL-1

$\Psi_t = \{[10+63t, 6, 4+32t]_2\}$ $|\Psi_t| = 34$ for $t \geq 2$; $\alpha = 1/2, \beta = -(2+16t)$;
$\Psi_t^i = \{[g_2(6+i, 4.2^i + t.2^{i+5}), 6+i, 4.2^i + t.2^{i+5}]_2\}$ $|\Psi_t^i| = 40$ for $t \geq 2$ and $i \geq 2$; $\alpha = 1/2^{i+1}, \beta = -d/2^{i+1}$;

$[34,3,8]_2$ $\xi = 8$; $4^2 2^6 1^{14}$;TOTAL – 1
$[34,3,16]_2$ $\xi = 8$; $3^2 3^6 1^{10}$; TOTAL – 1
$[34,4,8]_2$ $\xi = 8$; $6^2 2^4 1^{14}$; $5^2 2^4 1^6 1^{10}$; $6^2 1^4 1^6 1^{12}$; $4^2 2^4 3^6$; $6^2 2^4 1^6 1^8$; $4^2 3^6 1^8$; TOTAL – 6
$[34,4,16]_2$ $\xi = 8$; $6^2 4^4 1^6$; TOTAL – 1
$[34,5,8]_2$ $\xi = 8$; $9^2 1^6 1^{10}$; $9^2 1^4 2^6$; $10^2 1^6 1^8$; $10^2 1^{14}$; $10^2 2^4 1^6$; $11^2 3^4$; $5^2 3^4 2^6$; $6^2 2^4 1^6 1^8$; $7^2 3^4 1^8$; $6^2 4^4 1^6$; $7^2 5^4$; $6^2 3^4 1^{10}$; $7^2 2^4 1^{12}$; TOTAL – 13
$[34,5,16]_2$ $\xi = 8$; $15^2 1^4$; TOTAL – 1
$[34,6,8]_2$ $\xi = 8$; $6^2 4^4 1^6$; $10^2 2^4 1^6$; $11^2 3^4$; $12^2 1^4 1^6$; $13^2 2^4$; $7^2 5^4$; $11^2 2^6$; $15^2 1^4$; 17^2; $10^2 1^6 1^8$; $7^2 3^4 1^8$; TOTAL – 11
$[34,7,8]_2$ $\xi = 8$; 17^2; $15^2 1^4$; $11^2 3^4$; $13^2 2^4$; $7^2 5^4$; TOTAL – 5
$[34,8,8]_2$ $\xi = 8$; 17^2; TOTAL – 1

$\Psi_t = \{[14+63t, 6, 6+32t]_2\}$ $|\Psi_t| = 17$ for $t \geq 2$; $\alpha = 1/2, \beta = -(3+16t)$;
$\Psi_t^i = \{[g_2(6+i, 6.2^i + t.2^{i+5}), 6+i, 6.2^i + t.2^{i+5}]_2\}$ $|\Psi_t^i| = 20$ for $t \geq 2$ and $i \geq 2$; $\alpha = 1/2^{i+1}, \beta = -d/2^{i+1}$;

$[35,4,8]_2$ $\xi = 8$; $12^1 2^5 1^{13}$; $11^1 3^5 1^9$; TOTAL – 2
$[35,5,8]_2$ $\xi = 8$; $14^1 2^4 1^{13}$; $13^1 2^4 1^5 1^9$; $12^1 2^4 3^5$; $14^1 1^4 1^5 1^{12}$; $14^1 2^4 1^5 1^8$; $14^1 4^4 1^5$; $12^1 3^5 1^8$; TOTAL – 7
$[35,6,8]_2$ $\xi = 8$; $13^1 3^4 2^5$; $14^1 2^4 1^5 1^8$; $15^1 3^4 1^8$; $15^1 5^4$; $14^1 4^4 1^5$; $14^1 3^4 1^9$; $15^1 2^4 1^{12}$; TOTAL – 7
$[35,7,8]_2$ $\xi = 8$; $14^1 4^4 1^5$; $15^1 5^4$; $15^1 3^4 1^8$; TOTAL – 3
$[35,8,8]_2$ $\xi = 8$; $15^1 5^4$; TOTAL – 1
$[35,4,16]_2$ $\xi = 8$; $10^1 5^5$; TOTAL – 1

$\Psi_t = \{[17+63t, 6, 8+32t]_2\}$ $|\Psi_t| = 3$ *for* $t \geq 1$ $\alpha = 1/2, \beta = -(4+16t)$;
$\Psi_t^i = \{[g_2(6+i, 8.2^i + t.2^{i+5}), 6+i, 8.2^i + t.2^{i+5}]_2\}$ $|\Psi_t^i| = 3$ *for* $t \geq 1$*and* $i \geq 0$ $\alpha = 1/2^{i+1}, \beta = -d/2^{i+1}$;

$[20,2,8]_2$ $\xi = 8$; $2^4 1^{12}$; TOTAL – 1
$[20,3,8]_2$ $\xi = 8$; $3^4 1^8$; TOTAL – 1
$[20,4,8]_2$ $\xi = 8$; 5^4; TOTAL – 1

$\Psi_t = \{[22+63t, 6, 10+32t]_2\}$ $|\Psi_t| = 15$ *for* $t \geq 2$; $\alpha = 1/2, \beta = -(5+16t)$;
$\Psi_t^i = \{[g_2(6+i, 10.2^i + t.2^{i+5}), 6+i, 10.2^i + t.2^{i+5}]_2\}$ $|\Psi_t^i| = 21$ for $t \geq 2$ and $i \geq 2$; $\alpha = 1/2^{i+1}, \beta = -d/2^{i+1}$;

$[37, 4, 8]_2$ $\xi = 8$; $8^1 6^3 1^{11}$; TOTAL – 1
$[37, 5, 8]_2$ $\xi = 8$; $12^1 4^2 2^3 1^{11}$; $11^1 4^2 3^3 1^9$; $12^1 3^2 3^3 1^{10}$; $12^1 4^2 3^3 1^8$; $8^1 7^3 1^8$; TOTAL – 5
$[37, 6, 8]_2$ $\xi = 8$; $14^1 6^2 1^{11}$; $14^1 5^2 1^3 1^{10}$; $13^1 6^2 1^3 1^9$; $14^1 6^2 1^3 1^8$; $20^1 2^3 1^{11}$; $19^1 3^3 1^9$; $20^1 3^3 1^8$; $12^1 4^2 3^3 1^8$; TOTAL – 8
$[37, 7, 8]_2$ $\xi = 8$; $15^1 7^2 1^8$; $14^1 6^2 1^3 1^8$; $14^1 7^2 1^9$; $20^1 3^3 1^8$; $15^1 6^2 1^{10}$; TOTAL – 5
$[37, 8, 8]_2$ $\xi = 8$; $15^1 7^2 1^8$; TOTAL – 1
$[37, 4, 16]_2$ $\xi = 8$; $7^1 7^3 1^9$; TOTAL – 1

$\Psi_t = \{[25+63t, 6, 12+32t]_2\}$ $|\Psi_t| = 2$ for $t \geq 1$; $\alpha = 1/2, \beta = -(6+16t)$;
$\Psi_t^i = \{[g_2(6+i, 12.2^i + t.2^{i+5}), 6+i, 12.2^i + t.2^{i+5}]_2\}$ $|\Psi_t^i| = 2$ for $t \geq 1$ and $i \geq 0$; $\alpha = 1/2^{i+1}, \beta = -d/2^{i+1}$;

$[22, 3, 8]_2$ $\xi = 8$; $6^2 1^{10}$; TOTAL-1; $[22, 4, 8]_2$ $\xi = 8$; $7^2 1^8$; TOTAL - 1;

$\Psi_t = \{[29+63t, 6, 14+32t]_2\}$ $|\Psi_t| = 2$ *for* $t \geq 1$; $\alpha = 1/2, \beta = -(7+16t)$;
$\Psi_t^i = \{[g_2(6+i, 14.2^i + t.2^{i+5}), 6+i, 14.2^i + t.2^{i+5}]_2\}$ $|\Psi_t^i| = 2$ for $t \geq 1$ and $i \geq 0$; $\alpha = 1/2^{i+1}, \beta = -d/2^{i+1}$;

$[23, 4, 8]_2$ $\xi = 8$; $14^1 1^9$; TOTAL – 1; $[23, 5, 8]_2$ $\xi = 8$; $15^1 1^8$; TOTAL – 1;

$\Psi_t = \{[38+63t, 6, 18+32t]_2\}$ $|\Psi_t| = 43$ for $t \geq 2$; $\alpha = 1/2, \beta = -(9+16t)$;
$\Psi_t^i = \{[g_2(6+i, 18.2^i + t.2^{i+5}), 6+i, 18.2^i + t.2^{i+5}]_2\}$ $|\Psi_t^i| = 84$ for $t \geq 2$ and $i \geq 3$; $\alpha = 1/2^{i+1}, \beta = -d/2^{i+1}$;

$[41, 4, 8]_2$ $\xi = 8$; $8^1 4^3 3^7$; TOTAL – 1
$[41, 4, 16]_2$ $\xi = 8$; $6^1 6^3 2^5 1^7$; TOTAL – 1
$[41, 5, 8]_2$ $\xi = 8$; $12^1 4^2 3^7$; $8^1 6^3 2^4 1^7$; $10^1 4^2 2^3 2^5 1^7$; $7^1 7^3 2^4 1^5$; $8^1 4^2 6^3 1^7$; $9^1 4^2 3^3 3^5$; $12^1 4^2 2^3 2^4 1^7$; $12^1 2^2 2^3 2^6 1^7$; $11^1 3^2 3^3 1^4 1^5 1^6$; TOTAL – 9
$[41, 5, 16]_2$ $\xi = 8$; $11^1 4^2 3^3 2^4 1^5$; $10^1 5^2 5^3 1^6$; $12^1 8^2 2^3 1^7$; TOTAL – 3
$[41, 6, 8]_2$ $\xi = 8$; $14^1 4^2 2^6 1^7$; $12^1 4^2 2^3 2^4 1^7$; $12^1 6^2 2^5 1^7$; $12^1 3^2 3^3 2^4 1^6$; $13^1 5^2 1^3 1^4 1^5 1^6$; $14^1 6^2 2^4 1^7$; $20^1 2^3 2^4 1^7$; $10^1 6^2 4^3 1^7$; $11^1 6^2 1^3 3^5$; $11^1 4^2 3^3 2^4 1^5$; $12^1 8^2 2^3 1^7$; $13^1 6^2 1^3 2^4 1^5$; $12^1 4^2 3^3 3^4$; $12^1 7^2 3^3 1^6$; $14^1 10^2 1^7$; $20^1 4^2 2^3 1^7$; $19^1 3^3 2^4 1^5$; $14^1 5^2 1^3 2^4 1^6$; $14^1 9^2 1^3 1^6$; $14^1 6^2 1^3 3^4$; $14^1 3^2 1^3 3^6$; $22^1 6^2 1^7$; $18^1 2^2 4^3 1^7$; $18^1 2^3 2^5 1^7$; $18^1 3^3 1^4 2^5$; $20^1 3^7$; $20^1 3^3 3^4$; $8^1 7^3 3^4$; TOTAL–28
$[41, 6, 16]_2$ $\xi = 8$; $30^1 2^2 1^7$; TOTAL – 1
$[41, 7, 8]_2$ $\xi = 8$; $14^1 5^2 1^3 2^4 1^6$; $14^1 6^2 1^4 1^5 1^6$; $15^1 6^2 2^4 1^6$; $22^1 4^3 1^7$; $14^1 6^2 2^4 1^7$; $20^1 2^3 2^4 1^7$; $19^1 3^3 2^4 1^5$; $20^1 4^2 2^3 1^7$; $12^1 8^2 2^3 1^7$; $20^1 3^3 3^4$; $22^1 6^2 1^7$; $26^1 4^2 1^7$; $24^1 2^2 2^3 1^7$; $14^1 10^2 1^7$; $14^1 9^2 1^3 1^6$; $13^1 8^2 2^3 1^6$; $15^1 10^2 1^6$; $13^1 6^2 1^3 2^4 1^5$; $14^1 6^2 1^3 3^4$; $14^1 7^2 2^4 1^5$; $15^1 7^2 3^4$; $30^1 2^2 1^7$; $34^1 1^7$; $12^1 4^2 3^3 3^4$; $12^1 7^2 3^5$; $15^1 4^2 3^6$; TOTAL – 26
$[41, 8, 8]_2$ $\xi = 8$; $15^1 10^2 1^6$; $14^1 9^2 1^3 1^6$; $26^1 4^2 1^7$; $14^1 10^2 1^7$; $22^1 6^2 1^7$; $30^1 2^2 1^7$; $34^1 1^7$; $15^1 7^2 3^4$; $14^1 6^2 1^3 3^4$; $14^1 7^2 2^4 1^5$; $15^1 6^2 2^4 1^6$; $20^1 3^3 3^4$; TOTAL – 12
$[41, 9, 8]_2$ $\xi = 8$; $34^1 1^7$; $15^1 10^2 1^6$; $15^1 7^2 3^4$; TOTAL – 3

$\Psi_t = \{[41+63t, 6, 20+32t]_2\}$ $|\Psi_t| = 4$ for $t \geq 1$, $\alpha = -\frac{1}{2}, \beta = 10+16t$; $\alpha = 1/2, \beta = -(10+16t)$;
$\Psi_t^i = \{[g_2(6+i, 20.2^i + t.2^{i+5}), 6+i, 20.2^i + t.2^{i+5}]_2\}$ $|\Psi_t^i| = 4$ for $t \geq 1$ and $i \geq 0$; $\alpha = 1/2^{i+1}, \beta = -d/2^{i+1}$;

$[26, 3, 8]_2$ $\xi = 8$; $4^2 3^6$; TOTAL – 1
$[26, 4, 8]_2$ $\xi = 8$; $6^2 2^4 1^6$; TOTAL – 1
$[26, 5, 8]_2$ $\xi = 8$; $10^2 1^6$; $7^2 3^4$; TOTAL – 2

$\Psi_t = \{[45+63t, 6, 22+32t]_2\}$ $|\Psi_t| = 3$ for $t \geq 1$; $\alpha = 1/2, \beta = -(11+16t)$;
$\Psi_t^i = \{[g_2(6+i, 22.2^i + t.2^{i+5}), 6+i, 22.2^i + t.2^{i+5}]_2\}$ $|\Psi_t^i| = 3$ for $t \geq 1$ and $i \geq 0$; $\alpha = 1/2^{i+1}, \beta = -d/2^{i+1}$;

$[27, 4, 8]_2$ $\xi = 8$; $12^1 3^5$; TOTAL – 1
$[27, 5, 8]_2$ $\xi = 8$; $14^1 2^4 1^5$; TOTAL – 1
$[27, 6, 8]_2$ $\xi = 8$; $15^1 3^4$; TOTAL – 1

$\Psi_t = \{[53+63t, 6, 26+32t]_2\}$ $|\Psi_t| = 4$ for $t \geq 1$; $\alpha = 1/2, \beta = -(13+16t)$;
$\Psi_t^i = \{[g_2(6+i, 26.2^i + t.2^{i+5}), 6+i, 26.2^i + t.2^{i+5}]_2\}$ $|\Psi_t^i| = 5$ for $t \geq 1$ and $i \geq 1$; $\alpha = 1/2^{i+1}, \beta = -d/2^{i+1}$;

$[29, 4, 8]_2$ $\xi = 8$; $8^1 7^3$; TOTAL – 1
$[29, 5, 8]_2$ $\xi = 8$; $12^1 4^2 3^3$; TOTAL – 1
$[29, 6, 8]_2$ $\xi = 8$; $14^1 6^2 1^3$; $20^1 3^3$; TOTAL – 2

$[29,7,8]_2$ $\xi = 8$; $15^1 7^2$; TOTAL – 1

$\Psi_t = \{[13+40t, 4, 8+27t]_3\}$ $|\Psi_t| = 5$ for $t \geq 2$; $\alpha = 1, \beta = -d$
$\Psi_t^i = \{[g_3(4+i, 8.3^i + t.3^{i+3}), 4+i, 8.3^i + t.3^{i+3}]_3\}$ $|\Psi_t^i| = 6$ for $t \geq 2$ and $i \geq 0$; $\alpha = 1/3^i, \beta = -d/3^i$

$[31,3,9]_3$ $\xi = 9$; $12^1 1^{19}$; $11^1 2^{10}$; TOTAL – 2
$[31,4,9]_3$ $\xi = 9$; $12^1 1^9 1^{10}$; $13^1 2^9$; $13^1 1^{18}$; TOTAL – 3
$[31,5,9]_3$ $\xi = 9$; $13^1 2^9$; TOTAL – 1

$\Psi_t = \{[22+40t, 4, 14+27t]_3\}$ $|\Psi_t| = 7$ for $t \geq 0$; $\alpha = 1, \beta = -d$
$\Psi_t^i = \{[g_3(4+i, 14.3^i + t.3^{i+3}), 4+i, 14.3^i + t.3^{i+3}]_3\}$ $|\Psi_t^i| = 12$ for $t \geq 2$ and $i \geq 2$; $\alpha = 1/3^i, \beta = -d/3^i$

$[34,3,9]_3$ $\xi = 9$; $9^1 3^4 1^{13}$; TOTAL – 1
$[34,3,18]_3$ $\xi = 9$; $8^1 4^4 1^{10}$; TOTAL – 1
$[34,4,9]_3$ $\xi = 9$; $11^1 3^3 1^4 1^{10}$; $12^1 3^3 1^4 1^9$; $12^1 2^3 1^4 1^{12}$; $12^1 3^3 1^{13}$; $9^1 4^4 1^9$; TOTAL – 5
$[34,5,9]_3$ $\xi = 9$; $12^1 3^3 1^4 1^9$; $13^1 4^3 1^9$; $12^1 4^3 1^{10}$; $13^1 3^3 1^{12}$; TOTAL – 4
$[34,6,9]_3$ $\xi = 9$; $13^1 4^3 1^9$; TOTAL – 1

$\Psi_t = \{[25+40t, 4, 16+27t]_3\}$ $|\Psi_t| = 5$ for $t \geq 2$; $\alpha = 1, \beta = -d$
$\Psi_t^i = \{[g_3(4+i, 16.3^i + t.3^{i+3}), 4+i, 16.3^i + t.3^{i+3}]_3\}$ $|\Psi_t^i| = 13$ for $t \geq 2$ and $i \geq 3$; $\alpha = 1/3^i, \beta = -d/3^i$

$[35,3,18]_3$ $\xi = 9$; $12^2 1^{11}$; TOTAL – 1
$[35,4,9]_3$ $\xi = 9$; $17^1 4^2 1^{10}$; $18^1 3^2 1^{11}$; $13^2 1^9$; TOTAL – 3
$[35,4,18]_3$ $\xi = 9$; $18^1 4^2 1^9$; TOTAL – 1
$[35,5,9]_3$ $\xi = 9$; $24^1 1^2 1^9$; $23^1 1^2 1^{10}$; $24^1 1^{11}$; $18^1 4^2 1^9$; TOTAL – 4
$[35,6,9]_3$ $\xi = 4$; $26^1 1^9$; $24^1 1^2 1^9$; $25^1 1^{10}$; TOTAL – 3
$[35,7,9]_3$ $\xi = 9$; $26^1 1^9$; TOTAL – 1

$\Psi_t = \{[6+21t, 3, 4+16t]_4\}$ $|\Psi_t| = 4$ for $t \geq 2$; $\alpha = 1, \beta = -d$
$\Psi_t^i = \{[g_4(3+i, 4.4^i + t.4^{i+2}), 3+i, 4.4^i + t.4^{i+2}]_4\}$ $|\Psi_t^i| = 4$ for $t \geq 2$ and $i \geq 0$; $\alpha = 1/4^i, \beta = -d/4^i$

$[12,2,4]_4$ $\xi = 4$; $1^4 1^8$; TOTAL – 1
$[12,2,8]_4$ $\xi = 4$; 3^4; TOTAL – 1
$[12,3,4]_4$ $\xi = 4$; 3^4; TOTAL – 1
$[12,3,8]_4$ $\xi = 4$; 6^2; TOTAL – 1

$\Psi_t = \{[14+21t, 3, 10+16t]_4\}$ $|\Psi_t| = 6$ for $t \geq 2$; $\alpha = 1, \beta = -d$
$\Psi_t^i = \{[g_4(3+i, 10.4^i + t.4^{i+2}), 3+i, 10.4^i + t.4^{i+2}]_4\}$ $|\Psi_t^i| = 10$ for $t \geq 2$ and $i \geq 2$; $\alpha = 1/4^i, \beta = -d/4^i$

$[14,2,8]_4$ $\xi = 4$; $4^2 1^6$; TOTAL – 1
$[14,3,4]_4$ $\xi = 4$; $8^1 1^6$; $7^1 1^2 1^5$; $5^2 1^4$; TOTAL – 3
$[14,3,8]_4$ $\xi = 4$; $8^1 1^2 1^4$; 7^2; TOTAL – 2
$[14,4,4]_4$ $\xi = 4$; $8^1 1^2 1^4$; $10^1 1^4$; $9^1 1^5$; TOTAL – 3
$[14,5,4]_4$ $\xi = 4$; $10^1 1^4$; TOTAL – 1

$\Psi_t = \{[19+21t, 3, 14+16t]_4\}$ $|\Psi_t| = 1$ for $t \geq 1$; $\alpha = 1, \beta = -d$
$\Psi_t^i = \{[g_4(3+i, 14.4^i + t.4^{i+2}), 3+i, 14.4^i + t.4^{i+2}]_4\}$ $|\Psi_t^i| = 2$ for $t \geq 1$ and $i \geq 1$; $\alpha = 1/4^i, \beta = -d/4^i$

$[10,3,4]_4$ $\xi = 4$; $8^1 1^2$; TOTAL – 1;
$[10,4,4]_4$ $\xi = 4$; 10^1; TOTAL – 1;

Table 1: Parameters of the dual codes of projective Griesmer codes over $\mathbb{F}_2$ for $\alpha = 1, \beta = -d$

d/k	4	5	6
1–	[17,k,8;12]	[49,k,24;32]	[129,k,64;80]
2–	[10,k,4;8]	[34,k,16;24]	[98,k,48;64]
3–	[11,k,4;8]	[35,k,16;24]	[99,k,48;64]
4–	[4,k,0;4]	[20,k,8;16]	[68,k,32;48]
5–	[13,k,4;8]	[37,k,16;24]	[101,k,48;64]
6–	[6,k,0;4]	[22,k,8;16]	[70,k,32;48]
7–	[7,k,0;4]	[23,k,8;16]	[71,k,32;48]
8–		[8,k,0;8]	[40,k,16;32]
9–		[41,k,16;24]	[105,k,48;64]
10–		[26,k,8;16]	[74,k,32;48]
11–		[27,k,8;16]	[75,k,32;48]
12–		[12,k,0;8]	[44,k,16;32]
13–		[29,k,8;16]	[77,k,32;48]
14–		[14,k,0;8]	[46,k,16;32]
15–		[15,k,0;8]	[47,k,16;32]
16–			[16,k,0;16]
17–			[113,k,48;64]
18–			[82,k,32;48]
19–			[83,k,32;48]
20–			[52,k,16;32]
21–			[85,k,32;48]
22–			[54,k,16;32]
23–			[55,k,16;32]
24–			[24,k,0;16]
25–			[89,k,32;48]
26–			[58,k,16;32]
27–			[59,k,16;32]
28–			[28,k,0;16]
29–			[61,k,16;32]
30–			[30,k,0;16]
31–			[31,k,0;16]

Table 2: Number of the nonequivalent codes with given parameters over $\mathbb{F}_2$

$[n,k,d]/k$	1	2	3	4	5	6	7	8	9	10
$[49,k,8;16,24,32]$	0	0	0	2	65	603				
$[49,k,16;24,32]$	0	0	0	6	69	273				
$[49,k,24;32]$	0	0	0	1	1	1	0	0	0	0
$[34,k,8;16,24]$	0	0	1	6	13	11	5	1	0	0
$[34,k,16;24]$	0	0	1	1	1	0	0	0	0	0
$[35,k,8;16,24]$	0	0	0	2	7	7	3	1	0	0
$[35,k,16;24]$	0	0	0	1	0	0	0	0	0	0
$[20,k,8;16]$	0	1	1	1	0	0	0	0	0	0
$[37,k,8;16,24]$	0	0	0	1	5	8	5	1	0	0
$[37,k,16;24]$	0	0	0	1	0	0	0	0	0	0
$[22,k,8;16]$	0	0	0	1	0	0	0	0	0	0
$[23,k,8;16]$	0	0	0	1	1	0	0	0	0	0
$[8,k,0;8]$	1	0	0	0	0	0	0	0	0	0
$[41,k,8;16,24]$	0	0	0	1	9	28	26	12	3	0
$[41,k,16;24]$	0	0	0	1	3	1	0	0	0	0
$[26,k,8;16]$	0	0	1	1	2	0	0	0	0	0
$[27,k,8;16]$	0	0	0	1	1	1	0	0	0	0
$[12,k,0;8]$	0	1	0	0	0	0	0	0	0	0
$[29,k,8;16]$	0	0	0	1	1	2	1	0	0	0
$[14,k,0;8]$	0	0	1	0	0	0	0	0	0	0
$[15,k,0;8]$	0	0	0	1	0	0	0	0	0	0