On the ternary projective codes with dimensions 4 and 5

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Abstract

All ternary projective codes of dimension 4 and these of dimension 5 of lengths up to 15 are classified. Their automorphism groups and weight spectra are determined. The lest value of the covering radius of ternary codes of dimension 4 and lengths between 13 and 20 are computed.

I Preliminaries

In this work we investigate ternary projective codes of dimensions 4 and 5. The approach we have used is to classify all such codes of dimension 4 and the codes of dimension 5 and lengths up to 15 and to determine some of their basic characteristics like automorphism groups, weight spectra and covering radius.

All codes considered are ternary and linear. As usual, an $[n, k, d]$ code C is a k-dimensional subspace of the n-dimensional vector space F_q^n over the q-ary field F_q with a minimum Hamming distance d.

A k-by-n matrix G having as rows the vectors of a basis of C is called a *generator matrix* of C.

Let A_i denote the number of codewords of C of weight i. Then the numbers $A_0, \ldots, A_n$ are called the *weight spectrum* of the code C.

Let C_1 and C_2 be two linear $[n, k]_q$ codes. They are said to be equivalent if the codewords of C_2 can be obtained from the codewords of C_1 via a sequence of transformations of the following types:

(1) permutation of the set of coordinate positions;

(2) multiplication of the elements in a given position by a non-zero element of F_q;

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(3) application of a field automorphism to the elements in all coordinate positions.

An automorphism of a linear code C is a finite sequence of transformations of type (1)-(3), which maps each codeword of C onto a codeword of C. All the automorphisms of a code C form a group, which is called the *automorphism group* $Aut(C)$ of the code.

A *coset* of the code C defined by the vector $x \in F_q^n$ is the set $x + C = \{x + c \mid c \in C\}$. A *coset leader* of $x + C$ is a vector in $x + C$ of smallest weight.

The greatest of the distances between a vector from the n-dimensional vector space F_q^n over $GF(q)$ and the code C is called the *covering radius* $R(C)$ of C. The covering radius of a linear code is equivalent to the weight of the heaviest leader of its cosets.

The function $t_q[n, k]$ is defined as the least value of $R(C)$ when C runs over the class of all linear $[n, k]$ codes over F_q for a given q.

II Codes construction

The first step of our investigation is the classification of the ternary projective codes of dimensions 4 and 5. A classification of these codes with respect to their minimum distances using projective geometries can be found in [4]. In our work we have used a McKay approach [5] for isomorphism rejection.

Let us denote by M the $q^m \times \dfrac{q^m - 1}{q - 1}$ matrix of the codewords of the simplex code. To construct every projective code we use the fact that it is a punctured version of the corresponding simplex code. In other words the codewords of every projective code C are obtained by taking some fixed number of the columns of the matrix M. We will say that the code C is defined by these columns of M.

The main idea using McKay-type approach is to construct recursively new child codes from parent codes. In our case if a parent code is defined of n columns of the matrix M the child code will be defined by these columns plus a new column from M. As child codes will be accepted only those codes that pass a parent test and an isomorphism test.

In the parent test we need a canonical labelling [5] of the coordinates of the codes. The parent test can be passed by those child codes which last added coordinate is first in the canonical labelling or is in the same orbit with the first in the canonical labelling coordinate. From the child codes which passed the parents test we take only one representative from each class of equivalence.

The construction algorithm. Start from an empty set and recursively do the following. For a given code in the search tree, construct all possible child codes obtained by adding one coordinate. For each such child, carry out the parent test and, for those who survive the parent test, carry out isomorph rejection with the isomorphism test among those codes that come from the same parent.

To calculate the canonical labelling of the coordinates of the codes and the isomorphism test the algorithm from [2] is used. An advantage of the algorithm for our investigation is that we have to find an isomorphism only between the child of one parent.

III Classification results

In the list below we present ternary projective codes of dimension 4 of all lengths and of dimension 5 of lengths up to 15. For each length the common number of the nonequivalent codes are given. Then the numbers of codes having the corresponding minimum distance are written as powers of this minimum distance. The complete list with all determined properties of the codes can be found at http://www.moi.math.bas.bg/∼iliya.

k= 4

$n = 4$	1	1^1
$n = 5$	3	1^2, 2^1
$n = 6$	8	1^3, 2^5
$n = 7$	19	1^4, 2^{11}, 3^4
$n = 8$	44	1^4, 2^{16}, 3^{21}, 4^3
$n = 9$	91	1^3, 2^{16}, 3^{45}, 4^{26}, 5^1
$n = 10$	199	1^3, 2^{13}, 3^{55}, 4^{112}, 5^{15}, 6^1
$n = 11$	401	1^2, 2^{10}, 3^{46}, 4^{174}, 5^{165}, 6^4
$n = 12$	806	1^1, 2^6, 3^{33}, 4^{154}, 5^{448}, 6^{164}
$n = 13$	1504	1^1, 2^3, 3^{19}, 4^{102}, 5^{478}, 6^{843}, 7^{58}
$n = 14$	2659	1^1, 2^2, 3^9, 4^{53}, 5^{314}, 6^{1318}, 7^{950}, 8^{12}
$n = 15$	4304	2^1, 3^5, 4^{22}, 5^{151}, 6^{941}, 7^{2559}, 8^{623}, 9^2
$n = 16$	6472	3^2, 4^9, 5^{57}, 6^{439}, 7^{2310}, 8^{3478}, 9^{177}
$n = 17$	8846	4^3, 5^{19}, 6^{153}, 7^{1099}, 8^{4617}, 9^{2937}, 10^{18}
$n = 18$	11127	5^5, 6^{45}, 7^{356}, 8^{2454}, 9^{6799}, 10^{1466}, 11^2
$n = 19$	12723	6^{10}, 7^{89}, 8^{782}, 9^{4582}, 10^{6935}, 11^{324}, 12^1
$n = 20$	13358	7^{16}, 8^{178}, 9^{1514}, 10^{6893}, 11^{4722}, 12^{35}
$n = 21$	12723	8^{28}, 9^{328}, 10^{2526}, 11^{7860}, 12^{1981}
$n = 22$	11127	9^{47}, 10^{528}, 11^{3587}, 12^{6530}, 13^{435}
$n = 23$	8846	10^{68}, 11^{763}, 12^{4220}, 13^{3747}, 14^{48}
$n = 24$	6472	11^{91}, 12^{977}, 13^{3913}, 14^{1484}, 15^7
$n = 25$	4304	12^{114}, 13^{1074}, 14^{2764}, 15^{351}, 16^1
$n = 26$	2659	13^{127}, 14^{1014}, 15^{1462}, 16^{55}, 17^1
$n = 27$	1505	14^{127}, 15^{801}, 16^{569}, 17^7, 18^1
$n = 28$	807	15^{113}, 16^{520}, 17^{171}, 18^3
$n = 29$	402	16^{90}, 17^{274}, 18^{38}
$n = 30$	201	17^{66}, 18^{127}, 19^8
$n = 31$	94	18^{45}, 19^{47}, 20^2
$n = 32$	47	19^{26}, 20^{20}, 21^1
$n = 33$	23	20^{15}, 21^8
$n = 34$	12	21^9, 22^3
$n = 35$	6	22^5, 23^1

$n = 36$	4	23^3, 24^1
$n = 37$	2	24^2
$n = 38$	1	25^1
$n = 39$	1	26^1
$n = 40$	1	27^1
k= 5		
$n = 6$	4	1^3, 2^1
$n = 7$	15	1^8, 2^7
$n = 8$	61	1^{19}, 2^{39}, 3^3
$n = 9$	277	1^{44}, 2^{161}, 3^{71}, 4^1
$n = 10$	1439	1^{91}, 2^{525}, 3^{702}, 4^{120}, 5^1
$n = 11$	8858	1^{199}, 2^{1512}, 3^{3886}, 4^{3229}, 5^{31}, 6^1
$n = 12$	62311	1^{401}, 2^{4009}, 3^{14807}, 4^{36060}, 5^{7029}, 6^5
$n = 13$	459828	1^{806}, 2^{9796}, 3^{45215}, 4^{199735}, 5^{199532}, 6^{4744}
$n = 14$	3346151	1^{1504}, 2^{22016}, 3^{119254}, 4^{723620}, $5^{1891164}$, 6^{588357}, 7^{236}
$n = 15$	23246482	1^{2659}, 2^{45253}, 3^{279897}, $4^{2061818}$, $5^{8919464}$, $6^{11302236}$, 7^{635151}, 8^4

IV Least covering radius of some ternary linear codes of dimension 4

The last step of this investigation is the determination of some of the unknown values of the function $t_3[n, 4]$. At table with bounds on the function $t_3[n, k]$ for codes of lengths up to 27 is given in [1]. Later the values of $t_3[10, 4]$ and $t_3[12, 4]$ are determined in [6]. Here we extend the table from [1] for codes of lengths up to 40 and the row for dimension 4 is the following:

n	5	6	7	8	9	10	11	12	13
	1	1	2	2	3	4	4	5	5-6
n	14	15	16	17	18	19	20	21	22
	5-6	6-7	6-8	7-9	7-9	8-10	8-11	9-11	10-12
n	23	24	25	26	27	28	29	30	31
	10-13	11-13	11-14	12-15	12-15	13-16	13-17	14-17	15-18
n	32	33	34	35	36	37	38	39	40
	15-19	16-19	16-20	17-21	17-21	18-22	19-23	19-23	20-24

The determination of the covering radii of the ternary codes of dimensions 2 and 3 in [1] showed that for all lengths up to $\frac{q^m-1}{q-1}$ for $m = 2, 3$ the least covering radius is reached by a projective code. That is why we have first tested all projective codes of the given length. If there are no codes with covering radius equal to the lover bound for $t_3[n, k]$, we have to test codes with repeated coordinates. To determine how many repeated coordinates there are in the generator matrix of the code with the searched covering radius we have used the lower bound for concatenation of codes from [3]. Let C_1 be an $[n_1, k_1]$ and C_2 be an $[n_2, k_2]$ code with $k_1 \leq k_2$ and generator matrices G_1 and G_2 respectively. We define the generator matrix of the code C as $[G'_1|G_2]$, where G'_1 is G_1 with $k_2 - k_1$ rows of zeros added. Then C is an $[n_1 + n_2, k_2]$ code, for which the covering radius satisfies $R(C) \geq R(C_1) + R(C_2)$.

Let the $[n, 4]$ code C have repeated coordinates and let them be on the first s places of the generator matrix of C. Then we can consider C as a concatenation of codes C_1 which is an $[s, 1]$ code and C_2 which is an $[n - s, 4]$ projective code. The covering radius of C_1 is $R(C_1) = \lfloor \frac{2s}{3} \rfloor$ and for the code C_2 we take a code with the least covering radius $R(C_2) = t_3[n - s, 4]$. This way we can determine the maximum number of the repeated coordinates of C. Having classified all projective $[n - s, 4]$ codes we extend them with the necessary numbers of equivalent coordinates to get the codes C and then use the algorithm from [2] to get only the nonequivalent ones.

In the search for codes with a given covering radius we use the fact that if the code is in a systematic form, a vector of each coset can be found by generating all vectors of the form $(\underbrace{0, \ldots, 0}_{k}, a)$, $a \in F_3^{n-k}$. Then we only test words of this form and with $wt(a) \geq t_3[n, 4]$. If during the search we get a coset lieder with this weight we break the process and report that the code with the given covering radius is found. If we get a coset lieder with weight greater than $t_3[n, k]$ we stop the search and report that the covering radius of the code is greater than the searched value.

This way were determined the least values of the covering radius of ternary linear codes of dimension 4 and lengths between 13 and 20 and they are the following $t_3[13, 4] = t_3[14, 4] = \mathbf{6}$, $t_3[15, 4] = \mathbf{7}$, $t_3[16, 4] = t_3[17, 4] = \mathbf{8}$, $t_3[18, 4] = t_3[19, 4] = \mathbf{9}$, $t_3[20, 4] = \mathbf{10}$.

Remark 1. There is a unique projective $[19, 4]$ code with the least covering radius 9. It has the following weight enumerator $A_0 = 1, A_9 = 2, A_{11} = 12, A_{12} = 16, A_{13} = 30, A_{14} = 14, A_{15} = 2, A_{17} = 4$

Remark 2. In all cases where the exact values of the function $t_3[n, k]$ were determined (i.e. $13 \leq n \leq 20$) there are projective codes which have covering radius equal to the least one.

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