New upper bounds on the smallest size of a complete cap in the space $PG(3, q)$

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Outline

INTRODUCTION NOTATION

$PG(3, q)$ $\Rightarrow$ projective **space** of dimension 3 over Galois field F_q

n-**cap** $\Rightarrow$a set of n points no three of which are collinear
a line meeting a cap $\Rightarrow$ **tangent** or **bisecant**

bisecant $\Rightarrow$ a line intersecting a cap in two points

a **point** A of $PG(3, q)$ is **covered** by a cap $\Rightarrow$
A lies on a **bisecant** of the cap

complete cap $\Rightarrow$ all points of $PG(3, q)$ are covered
by bisecants of the cap
$\Rightarrow$ one may not add a new point to a complete cap

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INTRODUCTION NOTATION

$t_2(3, q) \Rightarrow$ the smallest size of a complete cap in $PG(3, q)$

HARD OPEN CLASSICAL PROBLEM: **1950 $\rightarrow$**
exact value or upper bound on $t_2(3, q)$

$\bar{t}_2(3, q) \Rightarrow$ the smallest **known** size of a complete cap in $PG(3, q)$
including computer search

$$t_2(3, q) \leq \bar{t}_2(3, q)$$

exact values of $t_2(3, q)$ for $q \leq 7$

G. Faina, S. Marcugini, A. Milani, F. Pambianco Ars Combinatoria **1998**

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KNOWN BOUNDS on $t_2(3, q)$

trivial lower bound $t_2(3, q) > \sqrt{2}q \quad \forall\, q$

upper bounds

$$t_2(3,q) \leq \begin{cases} 2q + t_2(2,q) & q \text{ even} \quad * \\ 3q & q \leq 17 \quad ** \\ 4q & q \leq 89 \quad ** \\ 11q & q < 30000 \quad *** \end{cases}$$

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Algorithm FOP

FOP $\Rightarrow$ Fixed Order of Points

Points of $PG(3, q)$ are ordered: $A_1, A_2, \ldots, A_{q^3+q^2+q+1}$

ALGORITHM $K^{(1)} = \{A_1\}, \quad K^{(2)} = \{A_1, A_2\},$

$$K^{(j+1)} = K^{(j)} \cup \{A_{m(j)}\},$$

$m(j)$ is the minimum index such that the corresponding point is not covered by $K^{(j)}$.

NEW UPPER BOUNDS on $t_2(3, q)$ Notation

Algorithm FOP with lexicographical order of points represented in homogenous coordinates

R $\Rightarrow$ a set of 400 prime in the interval $[3, 3109]$

$$\phi_{up}(q) = \frac{1}{\ln(0.2 \cdot q)} + 0.7$$

$$\theta_{up}(q) = \frac{1}{2\ln(0.01 \cdot q)} + 0.6$$

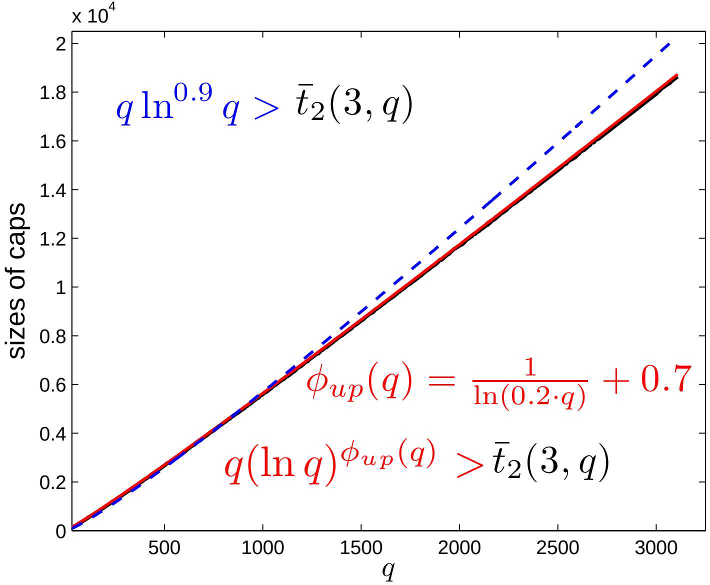
NEW UPPER BOUNDS on $t_2(3, q)$

Theorem

$$t_2(3,q) \le \begin{cases} 5q & \textit{if} \quad q \le 223 \\ 6q & \textit{if} \quad q \le 3109 \\ q \ln q & \textit{if} \quad 127 \le q \le 3109 \\ q \ln^{0.9} q & \textit{if} \quad 787 \le q \le 3109 \\ \theta_{up}(q) q \ln q & \textit{if} \quad 101 \le q \le 3109 \\ q \ln^{\phi_{up}(q)} q & \textit{if} \quad q \le 3109 \end{cases}$$

$q \in R$

NEW bounds based on a DEGREE OF LOGARITHM of q



on DECREASING DEGREE OF LOGARITHM of q

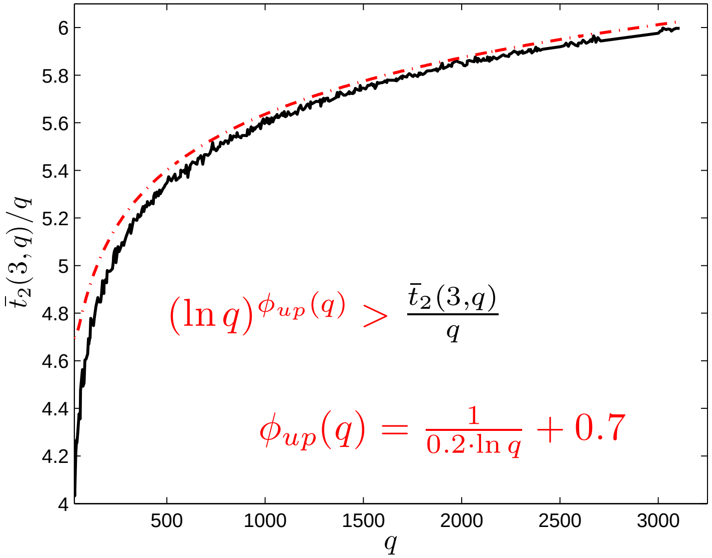
CDL-bound $\Rightarrow$ upper bound with a constant degree of $\log q$

DDL-bound $\Rightarrow$ upper bound with a decreasing degree of $\log q$

DDL-bounds are more convenient

DDL-bound and computer results $\bar{t}_2(3, q)$ almost coalesce but always DDL-bound $> \bar{t}_2(3, q)$

NEW bound based on a DECREASING DEGREE OF ln q



Thank you Spasibo

Mille grazie

Premnogo blagodarya

!‘Muchas gracias

Toda raba

Merci beaucoup

Dankeschön

Dank u wel

Domo arigato