Local distributions of q-ary eigenfunctions and of q-ary perfect colorings

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History

📕 *A. Yu. Vasil'eva,* Local spectra of perfect binary codes, Discrete Applied Mathematics, vol. 135, n.1-3 pp. 301-307, 2004 (Translated from, Discretn. anal. issled. oper. Ser.1 1999. V.6, No.1, 3-11)

📕 *A.Yu. Vasil'eva,* Local and Interweight Spectra of Completely Regular Codes and of Perfect Colourings, Probl. Inform. Transm. 45(2), 151-157 (2009). (Translated from Probl. Peredachi Inf., 45(2) (2009), P. 84-90.)

📕 *A.Yu. Vasil'eva,* Local Distribution and Reconstruction of Hypercube Eigenfunctions, Probl. Inform. Transm. ,V. 49(1), pp. 32-39 (2013). (Translated from Probl. Peredachi Inf., 49(1). P. 37-45 (2013))

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📕 *Soohak Choi, Jong Y. Hyun, Hyun K. Kim,* Local duality theorem for q-ary 1-perfect codes, Designs, Codes and Cryptography, May 2012, DOI 10.1007/s10623-012-9683-5

Notations

$$\mathbf{F}_q = \{0, 1, \ldots, q-1\}, \quad \mathbf{F}_q^n = \mathbf{F}_q \times \ldots \times \mathbf{F}_q$$

$$s(\alpha) = \{i \; : \; \alpha_i \neq 0\}, \quad \alpha \in \mathbf{F}_q^n$$

$$W_i(\alpha) = \{\beta \in \mathbf{F}_q^n \; : \; wt(\beta - \alpha) = i\}$$

$$I \subseteq \{1, \ldots, n\}, \quad \bar{I} = \{1, \ldots, n\} \backslash I$$

$$\Gamma_I(\alpha) = \{\beta \in \mathbf{F}_q^n \; : \; \beta_i = \alpha_i \; \forall \; i \notin I\},$$

then $\Gamma_I(\alpha)$ is said to be a $(n - |I|)$-dimensional face.

$$\langle \alpha, \beta \rangle = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n(\mod q), \quad \alpha, \beta \in \mathbf{F}_q^n$$

Eigenfunctions: definition

$$V = \left\{ f : F_q^n \longrightarrow \mathbb{C} \right\}$$

$$f \leftrightarrow (f(0, \ldots, 0), f(0, \ldots, 0, 1), \ldots, f(q-1, \ldots, q-1))^T$$

$$D - q^n \times q^n, \quad D_{\alpha, \beta} = \begin{cases} 1, & \rho(\alpha, \beta) = 1 \\ 0, & other \end{cases} \quad - \text{ adjacency matrix of } \mathbf{F}_q^n.$$

f is the eigenfunction of $\mathbf{F}^n$ with the eigenvalue λ (or λ-function) if

$$Df = \lambda f.$$

In other words, $\sum_{\beta \in W_1(\alpha)} f(\beta) = \lambda f(\alpha), \quad \forall \alpha \in \mathbf{F}_q^n.$
The eigenvalues of the graph of n-dimensional q-ary hypercube are equal to
$\lambda_i = (q-1)n - qi, \ i = 0, 1, \ldots, n$

Eigenfunctions and Fourier transform

$$\xi = e^{2\pi i/q}$$

A function

$$\varphi^\beta(\alpha) = \xi^{\langle \alpha, \beta \rangle}, \quad \alpha, \beta \in \mathbf{F}_q^n,$$

is called the character. The characters φ^β, $\beta \in \mathbf{F}_q^n$, forms the orthogonal basis of the vector space V. Define Fourier transform $\widehat{f}$ of a function f as follows:

$$\widehat{f(\alpha)} = \sum_{\beta \in \mathbf{F}_q^n} f(\beta)\varphi^\beta(\alpha) = \sum_{\beta \in \mathbf{F}_q^n} f(\beta)\xi^{\langle \alpha, \beta \rangle}, \quad \alpha \in \mathbf{F}_q^n.$$

In terms of Fourier transform f is the eigenfunction of $\mathbf{F}^n$ with the eigenvalue λ_i iff

$$\widehat{f(\alpha)} = 0 \quad \forall \ \alpha \notin W_i(\mathbf{0})$$

Eigenfunctions: Local distributions

By definition, put

$$v_j^{I,f}(\alpha) = \sum_{\beta \in \Gamma_I(\alpha) \bigcap W_j(\alpha)} f(\beta),$$

the vector $v^{I,f}(\alpha) = (v_0^{I,f}(\alpha), \ldots, v_{|I|}^{I,f}(\alpha))$ is called the local distribution of the function f in the face $\Gamma_I(\alpha)$ with respect to the vertex α, or shortly (I, α)-local distribution of f. We say that

$$g_f^{I,\alpha}(x,y) = \sum_{j=0}^{k} v_j^{I,f}(\alpha) y^j x^{k-j} = \sum_{\beta \in \Gamma_I(\alpha)} f(\beta) y^{|s(\beta)|} x^{|I|-|s(\beta)|}$$

is a local weight enumerator.

Eigenfunctions: Theorem

Theorem 1

Let λ be an eigenvalue of $\mathbf{F}_q^n$, f be a λ-function, $h = \frac{(q-1)n - \lambda}{q}$ and $\alpha \in \mathbf{F}_q^n$. Then

$$(x + (q-1)y)^{h - |\bar{I}|} g_f^{\bar{I}, \alpha}(x, y) = (x' + (q-1)y')^{h - |I|} g_f^{I, \alpha}(x', y'),$$

where $x' = x + (q-2)y$, $y' = -y$.

Rewrite the formula:

$$g_f^{\bar{I}}(x, y) = (x - y)^{h - |I|}(x + (q-1)y)^{|\bar{I}| - h} g_f^{I}(x + (q-2)y, -y)$$

Perfect colorings

The partition $C = (C_1, \ldots, C_r)$ of $\mathbf{F}_q^n$ is called a perfect coloring (or an equitable partition) with the parameter matrix $S = (s_{ij})_{i,j=1,\ldots,r}$ if for any $i, j \in \{1, \ldots, r\}$ and any vertex $\alpha \in C_i$ the number of vertices $\beta \in C_j$ at distance 1 from α is equal to s_{ij}. First present a perfect r-coloring by $(0, 1)$-matrix C of size $q^n \times r$ that defined as follows: each row has only one nonzero position that marks the color of the corresponding vertex. Then the coloring is perfect iff

$$DC = CS,$$

where D is the adjacency matrix of $\mathbf{F}_q^n$.
If the parameter matrix S is three-diagonal then C_0 (and C_r) is completely regular code.

Perfect colorings: local distributions

Define a local distribution of a coloring as local distributions of characteristic functions of the colors. More precisely, a local distribution of the coloring C in the face $\Gamma_I(\alpha)$ with respect to the vertex α is $(r \times |I|)$-matrix

$$
v^{I,C}(\alpha) = \begin{pmatrix} v_0^{I,C_1}(\alpha) & \ldots & v_{|I|}^{I,C_1}(\alpha) \\ \vdots & & \vdots \\ v_0^{I,C_r}(\alpha) & \ldots & v_{|I|}^{I,C_r}(\alpha) \end{pmatrix},
$$

where $v_j^{I,C_i}(\alpha) = |C_i \bigcap W_j(\alpha) \bigcap \Gamma_I(\alpha)|$, $i = 1, \ldots, r$, $j = 0, \ldots, |I|$. A vector-function

$$
g_C^{I,\alpha}(x,y) = (g_{C_1}^{I,\alpha}(x,y), \ldots, g_{C_r}^{I,\alpha}(x,y))
$$

is called the local weight enumerator of the coloring C in the face $\Gamma_I(\alpha)$ with respect to the vertex α.

Perfect colorings: Theorem

Theorem 2

Let $C = (C_1, \ldots, C_r)$ be a perfect coloring of $\mathbf{F}_q^n$ with parameter matrix S and $\alpha \in \mathbf{F}_q^n$. Put $h(S) = \frac{(q-1)nE - S}{q}$. Then

$$(x + (q-1)y)^{h(S) - |\bar{I}|E} g_C^{\bar{I}, \alpha}(x, y) = (x' + (q-1)y')^{h(S) - |I|E} g_C^{I, \alpha}(x', y'),$$

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$$(x + (q-1)y)^{h - |\bar{I}|} g_f^{\bar{I}, \alpha}(x, y) = (x' + (q-1)y')^{h - |I|} g_f^{I, \alpha}(x', y'),$$

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📕 *Jong Y. Hyun,* Local duality for equitable partitions of a Hamming space J. Comb. Theor. 119(2):7 (2012)

📕 *D.S. Krotov,* On weight distributions of perfect colorings and completely regular codes, Designs, Codes and Cryptography, V. 61, I. 3 , pp 315-329 (2011)

Thank you for your attention!