Two types of upper bounds on the smallest size of a complete arc in the plane $PG(2, q)$

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Outline

INTRODUCTION NOTATION

$PG(2,q)$ $\Rightarrow$ projective **space** of dimension 2 over Galois field F_q

n-**arc** $\Rightarrow$ a set of n points no three of which are collinear
a line meeting an arc $\Rightarrow$ **tangent** or **bisecant**

bisecant $\Rightarrow$ a line intersecting an arc in two points

a **point** A of $PG(2,q)$ is **covered** by an arc $\Rightarrow$
A lies on a **bisecant** of the arc

complete arc $\Rightarrow$ all points of $PG(2,q)$ are covered
by bisecants of the arc
$\Rightarrow$ one may not add a new point to a complete arc

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$t_2(2, q) \Rightarrow$ the smallest size of a complete arc in $PG(2, q)$

HARD OPEN CLASSICAL PROBLEM: **1950 $\rightarrow$**
exact value or upper bound on $t_2(2, q)$

$\bar{t}_2(2, q) \Rightarrow$ the smallest **known** size of a complete arc in $PG(2, q)$ including computer search

$$t_2(2, q) \leq \bar{t}_2(2, q)$$

exact values of $t_2(2, q)$ for $q \leq 32$

for $q = 31, 32$ D. Bartoli, A. Milani, S. Marcugini, F. Pambianco, *Journal of Geometry*, to appear

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KNOWN BOUNDS on $t_2(2, q)$

lower bound for $q = p^3$ O. Polverino **1999**

$$t_2(2,q) > \begin{cases} \sqrt{2q}+1 & \forall\, q \\ \sqrt{3q}+\frac{1}{2} & q = p^h,\ h = 1,2,3 \end{cases}$$

CDL-bound $\Rightarrow$ upper bound with a constant degree of $\log q$

DDL-bound $\Rightarrow$ upper bound with a decreasing degree of $\log q$

DDL-bounds are more convenient

KNOWN BOUNDS with a DEGREE of LOGARITHM

CDL-bound $t_2(2,q) \le d\sqrt{q}\log^c q, \quad c \le 300$

c, d constants independent of q J.H. Kim, V. Vu **2003**

theoretical bound; probabilistic methods

CDL-bound $t_2(2,q) < \sqrt{q}\ln^{0.73} q$

DDL-bound $t_2(2,q) < 0.7\sqrt{q}(\ln q)^{\frac{1}{\ln q}+0.78}$

estimates of computer results for all $q \le 13627$

D. Bartoli, A.A. Davydov, G. Faina, S. Marcugini, F. Pambianco **2005–2013**

small arcs + BDFMP = big love

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NEW DDL-bound with a DECREASING DEGREE of ln q

Theorem

DDL-bound $$t_2(2, q) < 0.6\sqrt{q}(\ln q)^{\varphi(q)}$$

$$\varphi(q) = \frac{1.5}{\ln q} + 0.8$$

all $q \leq 49727$, *all prime* $q \leq 89113$

100 sporadic prime q *with* $90001 \leq q \leq 350003$

estimates of computer results in HUGE region
ALL prime q WITHOUT GAPS

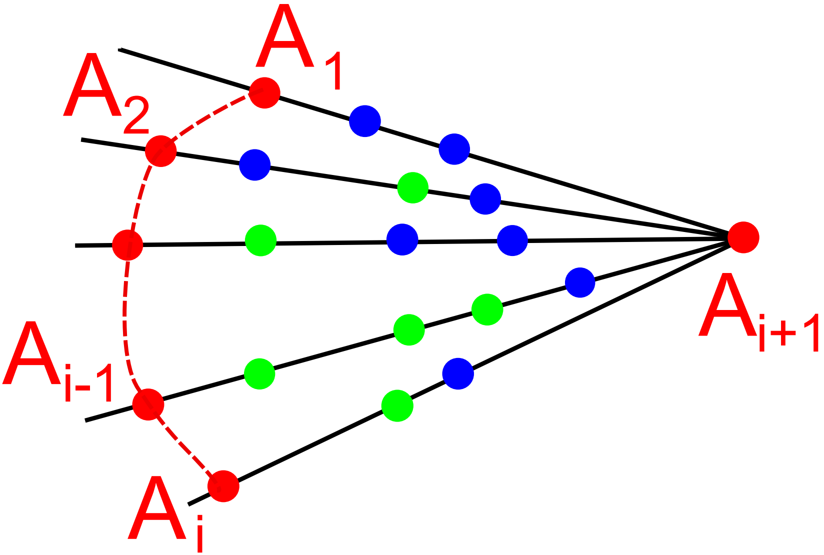
Randomized greedy algorithms

New arcs have been obtained using resources of Multipurpose Computing Complex of National Research Centre "Kurchatov Institute"

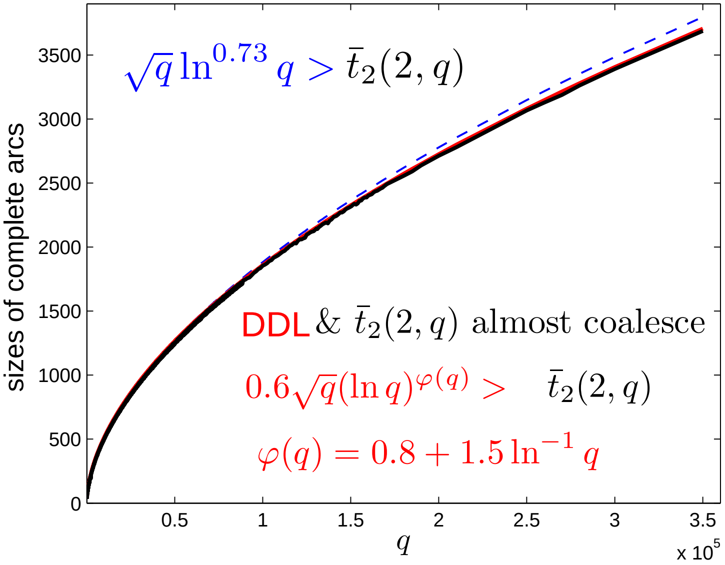
A greedy algorithm is an algorithm that makes the *locally optimal choice* at each stage with the hope of finding a global optimum or, at least, a global "good" solution.

A randomized greedy algorithm executes some stages in a random manner without the local optimum.

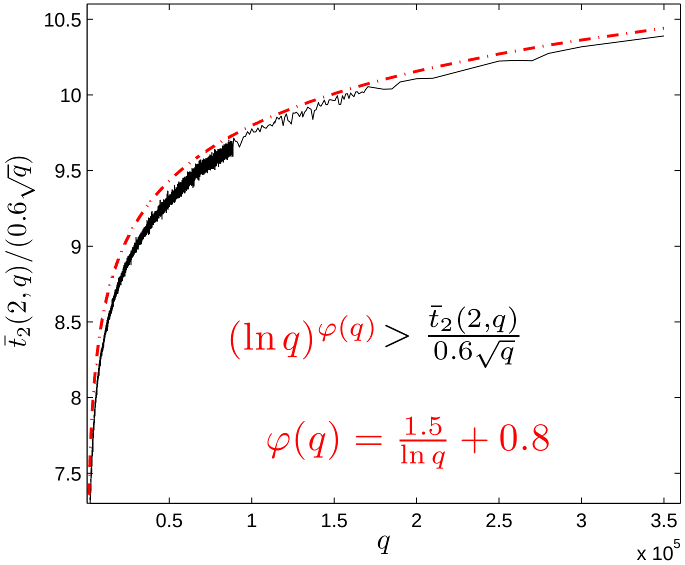
One step of greedy algorithm



DECREASING-DEGREE-LOGARITHM-bound > Arc's Size



$(\ln q)^{\varphi(q)} > \bar{t}_2(2, q)/(0.6\sqrt{q})$



Assumption of a uniform distribution

$\theta = q^2 + q + 1 \Rightarrow$ the number of points in $PG(2,q)$

$NONcov_i \Rightarrow$ the number of noncovered points of $PG(2,q)$ after the i-th step of the algorithm.

Assumption 1

Covered and noncovered points of the plane are distributed uniformly. The ***proportion*** $\frac{NONcov_i}{\theta}$ *of noncovered points is the* ***probability*** *that a random point of the plane is noncovered on the $i+1$-th step.*

PROBABILISTIC UPPER BOUNDS

Theorem

Under Assumption 1,

$$2\sqrt{q}\ln^{0.5} q > B(q) > t_2(2, q).$$

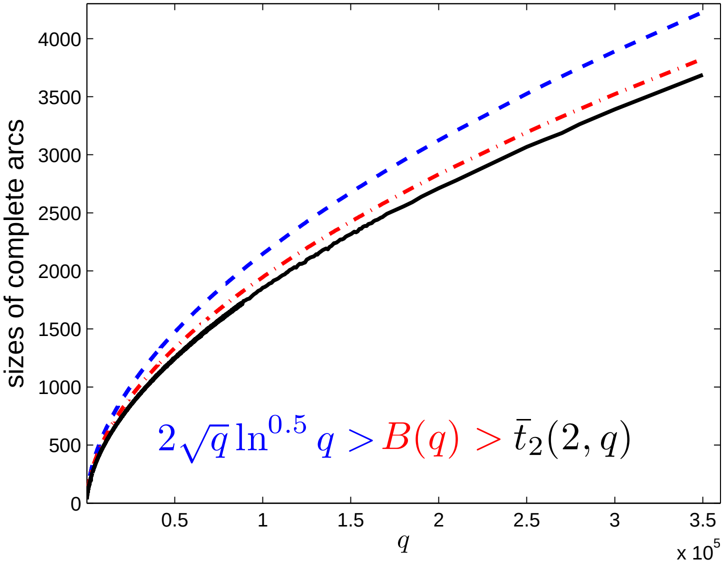
$B(q) = x$ *is the solution of equation*

$$\prod_{i=1}^{x}\left(1-\frac{i}{q}\right) = \frac{1}{\theta}\left(1+\sum_{j=1}^{x}\prod_{i=j}^{x}\left(1-\frac{i}{q}\right)\right)$$

assumption on uniform distribution + theoretical way

$2\sqrt{q}\ln^{0.5} q \Rightarrow$ rough estimate of $B(q)$

Probabilistic upper bounds



everything will be fine

All above mentioned new upper bounds are confirmed in huge region of q.

Conjecture

All above mentioned new upper bounds hold for all $\boldsymbol{q}$.

Thank you Spasibo

Mille grazie

Premnogo blagodarya

!‘Muchas gracias

Toda raba

Merci beaucoup

Dankeschön

Dank u wel

Domo arigato