An analogue of the Pless symmetry codes

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Introduction

Let C be a self-dual $[n, k, d]$- code over $\mathbb{F}_q$.

Type I	C is 2-divisible or even and $q = 2$
Type II	C is 4-divisible or doubly even and $q = 2$
Type III	C is 3-divisible and $q = 3$
Type IV	C is 2-divisible and $q = 4$

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In 1973 C.L. Mallows and N.J.A. Sloane proved that the minimum distance d of a self-dual $[n, k, d]$-code satisfies

Type I	$d \leq 2\left\lfloor \frac{n}{8} \right\rfloor + 2$
Type II	$d \leq 4\left\lfloor \frac{n}{24} \right\rfloor + 4, \text{ if } n \not\equiv 22 \mod 24$ $d \leq 4\left\lfloor \frac{n}{24} \right\rfloor + 6, \text{ if } n \equiv 22 \mod 24$
Type III	$d \leq 3\left\lfloor \frac{n}{12} \right\rfloor + 3$
Type IV	$d \leq 2\left\lfloor \frac{n}{6} \right\rfloor + 2$

Codes reaching the bound are called **Extremal**.

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Also the extended quadratic residue codes $XQR(p)$ of length $p+1$, whenever p prime

$$p \equiv \pm 1 \pmod{12},$$

define a series of self-dual ternary codes of high minimum distance.
In fact for small values of p both families define extremal codes.

The known extremal ternary codes of length $12n$.

Length n	$S(\frac{n}{2}-1)$	$XQR(n-1)$	Extremal distance	Partial Classification*
12	6		6	✓
24	9	9	9	✓
36	12	–	12	$o(\sigma) \geq 5$
48	15	15	15	$o(\sigma) \geq 5$
60	18	18	18	$o(\sigma) \geq 11$
72	–	18	21	No extremal

$*\ \sigma \in \mathrm{Aut}(C)$ of prime order.

Pless Construction

Given p an odd prime with $p \equiv -1 \pmod 6$. It is defined a matrix $S_p \in \mathbb{F}_3^{(p+1)\times(p+1)}$ by

$$S_p := \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ \chi(-1) & \chi(0) & \chi(1) & \cdots & \chi(p-1) \\ \chi(-1) & \chi(p-1) & \chi(0) & \cdots & \chi(p-2) \\ \vdots & \ddots & \cdots & & \\ \chi(-1) & \chi(1) & \chi(2) & \cdots & \chi(0) \end{pmatrix},$$

where

$$\chi(a) := \left(\frac{a}{p}\right) := \begin{cases} 0 & , \ p \mid a \\ 1 & , \ a \text{ is a quadratic residue mod } p, p \nmid a \\ -1 & , \ \text{otherwise} \end{cases}.$$

Then the code generated by the matrix $(\ I_{(p+1)} \mid S_p \ )$ is a self-dual $[2(p+1), p+1]-$code over $GF(3)$.

Definitions

Let K be a field, $n \in \mathbb{N}$. Then the **monomial group**

$$\mathrm{Mon}_n(K^*) \cong (K^*)^n : S_n \leq \mathrm{GL}_n(K),$$

the group of monomial $n \times n$-matrices over K, is the semidirect product of the subgroup $(K^*)^n$ of diagonal matrices in $\mathrm{GL}_n(K)$ with the group of permutation matrices.

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The **monomial automorphism group** of a code $C \leq K^n$ is

$$\mathrm{Aut}(C) := \{g \in \mathrm{Mon}_n(K^*) \mid Cg = C\}.$$

The idea to construct good self-dual codes is to investigate codes that are invariant under a given subgroup G of $\mathrm{Mon}_n(K^*)$. A very fruitful source are monomial representations, for some prime p, of $G = \mathrm{SL}_2(p)$.

Monomial Representations

Explicit generator matrices for the Pless codes may be obtained from the endomorphism ring of a monomial representation. Let p be an odd prime and

$$G := \mathrm{SL}_2(p) := \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \mathbb{F}_p^{2\times 2} \mid ad - bc = 1 \right\},$$

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$$B := \left\{ \begin{pmatrix} a & 0 \\ b & d \end{pmatrix} \in \mathrm{SL}_2(p) \right\} = \left\langle h_1 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \zeta := \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \right\rangle .$$

Then $B \cong (\mathbb{F}_p, +) : \mathbb{F}_p^*$, $[\mathrm{SL}_2(p) : B] = p + 1$ and $Z(B) = Z(\mathrm{SL}_2(p)) = \langle \zeta^{(p-1)/2} \rangle = \{\pm I_2\}$. The mapping $\lambda : B \to K^*; h_1 \mapsto 1, \zeta \mapsto -1$ is a linear character of B with kernel $\langle h_1, \zeta^2 \rangle$. It hence defines a monomial representation $\Delta = \lambda^G : G \to \mathrm{Mon}_{p+1}(K^*)$ with

$$\Delta(G) \cong \begin{cases} \mathrm{SL}_2(p) & p \equiv 1 \pmod 4 \\ \mathrm{PSL}_2(p) & p \equiv 3 \pmod 4 \end{cases}$$

Endomorphism ring of Δ

Then we may highlight the following:

i. $\mathrm{SL}_2(p) = B \dot\cup BwB$, $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

ii. A right transversal of B in $\mathrm{SL}_2(p)$ is $[1, wh_x : x \in \mathbb{F}_p]$ where $h_x := h_1^x$.

iii. (I_{p+1}, S_p) is the Schur basis of

$$\mathrm{End}(\Delta) := \{X \in K^{p+1\times p+1} | X\Delta(g) = \Delta(g)X \text{ for all } g \in G\}.$$

iv. $S_p^2 = \left(\frac{-1}{p}\right) p$ and $S_p S_p^{tr} = p$.

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To construct monomial representations of degree $2(p+1)$ we consider the group

$$\mathcal{G}(p) := \left\langle \begin{pmatrix} \Delta(g) & 0 \\ 0 & \Delta(g) \end{pmatrix}, Z := \begin{pmatrix} 0 & I_{p+1} \\ jI_{p+1} & 0 \end{pmatrix} \middle| g \in SL_2(p) \right\rangle \leq \mathrm{Mon}_{2(p+1)}(K^*)$$

where $j = -\left(\frac{-1}{p}\right) = \begin{cases} 1 & , p \equiv 3 \pmod 4 \\ -1 & , p \equiv 1 \pmod 4. \end{cases}$

Endomorphism Ring

Then we conclude

1. $\mathcal{G}(p) \cong \begin{cases} C_4 \times PSL_2(p) & , \ p \equiv 1 \pmod 4 \\ C_2 \times SL_2(p) & , \ p \equiv 3 \pmod 4 \end{cases}$, is contained in the automorphism group of the Pless code $\mathcal{P}(p)$.

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II. $\mathrm{End}(\mathcal{G}(p)) = \left\{ \begin{pmatrix} A & B \\ jB & A \end{pmatrix} \middle| A, B \in \mathrm{End}(\Delta) \right\}$ is generated by

$$I_{2(p+1)}, X := \begin{pmatrix} S_p & 0 \\ 0 & S_p \end{pmatrix}, Y := \begin{pmatrix} 0 & I_{p+1} \\ jI_{p+1} & 0 \end{pmatrix}, XY = \begin{pmatrix} 0 & S_p \\ jS_p & 0 \end{pmatrix}$$

with $X^2 = -jp$, $Y^2 = j$, $XY = YX$, $(XY)^2 = -p$.

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III. If $K = \mathbb{F}_3$ and $p \equiv -1 \pmod 3$ then $(I_{2(p+1)} - XY)^2 = 0$ and the rows of

$$I_{2(p+1)} - XY = \begin{pmatrix} I_{p+1} & S_p \\ jS_p & I_{p+1} \end{pmatrix}$$

span the Pless code $\mathcal{P}(p)$.

Generalized Code

Definition

Let $K = \mathbb{F}_q$ be the finite field with q elements and assume that there is some $a \in K^*$ such that $a^2 = -p$. Then we put $P_q(p) := aI_{2(p+1)} + XY \in \mathrm{End}(\mathcal{G}(p))$ and define the **generalized Pless code** $\mathcal{P}_q(p) \leq K^{2(p+1)}$ to be the code spanned by the rows of $P_q(p)$.

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Theorem

Let $a \in \mathbb{F}_q^*$ such that $a^2 = -p$. The code $\mathcal{P}_q(p)$ has generator matrix $(aI_{p+1}|P)$ and is a self-dual code in $\mathbb{F}_q^{2(p+1)}$. The sum of the first two rows of this matrix has weight $(p+7)/2$ if q is odd and 4 if q is even. The group $\mathcal{G}(p)$ is a subgroup of $\mathrm{Aut}(\mathcal{P}_q(p))$.

In particular $\mathcal{P}_3(p)$ is the Pless symmetry code $\mathcal{P}(p)$ as given in [5].

Generalized Code

Minimum distance of the Pless codes computed with MAGMA.

p	5	11	17	23	29	41	47
$2(p+1)$	12	24	36	48	60	84	96
$d(\mathcal{P}_3(p))$	6	9	12	15	18	21	24
$\mathrm{Aut}(\mathcal{P}_3(p))$	$2.M_{12}$	$\mathcal{G}(11).2$	$\mathcal{G}(17).2$	$\mathcal{G}(23).2$	$\mathcal{G}(29).2$	$\geq \mathcal{G}(41)$	$\geq \mathcal{G}(47)$

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For $q = 5, 7,$ and 11 we computed $d(\mathcal{P}_q(p))$ with MAGMA:

(p,q)	$(11,5)$	$(19,5)$	$(29,5)$	$(31,5)$	$(3,7)$	$(5,7)$	$(13,7)$
$2(p+1)$	12	40	60	64	8	12	28
$d(\mathcal{P}_q(p))$	9	13	18	18	4	6	10

(p,q)	$(17,7)$	$(19,7)$	$(7,11)$	$(13,11)$	$(17,11)$	$(19,11)$
$2(p+1)$	36	40	16	28	36	40
$d(\mathcal{P}_q(p))$	12	13	7	10	12	13

Endomorphisms of monomial representations

We now construct for odd primes q and p a prime number such that $p-1 \equiv 4 \pmod 8$ a monomial representation of $\Delta' : \mathrm{SL}_2(p) \to \mathrm{Mon}_{2(p+1)}(\mathbb{F}_q^*)$.

$$B^{(2)} := \left\{ \begin{pmatrix} a^2 & 0 \\ b & a^{-2} \end{pmatrix} \middle| a \in \mathbb{F}_p^*, b \in \mathbb{F}_p \right\} \leq \mathrm{SL}_2(p),$$

of index $[\mathrm{SL}_2(p) : B^{(2)}] = 2(p+1)$ with an unique linear representation

$$\gamma : B^{(2)} \to \mathbb{F}_q^*, \ \gamma\left(\begin{pmatrix} a^2 & 0 \\ b & a^{-2} \end{pmatrix} \right) = \left(\frac{a}{p} \right).$$

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Then $\Delta' := \gamma_{B^{(2)}}^{\mathrm{SL}_2(p)}$ is a faithful monomial representation of degree $2(p+1)$.

Taking $w := \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$, as before, we obtain explicit matrices.

Endomorphisms of monomial representations

By assumption $2 \in \mathbb{F}_p^* \setminus (\mathbb{F}_p^*)^2$. Put $\epsilon := \mathrm{Diag}(2, 2^{-1})$. Then $B = B^{(2)} \dot\cup B^{(2)}\epsilon$ and

$$\mathrm{SL}_2(p) = B \dot\cup BwB = B^{(2)} \dot\cup B^{(2)}wB^{(2)} \dot\cup B^{(2)}\epsilon \dot\cup B^{(2)}\epsilon w B^{(2)}$$

and a right transversal is given by $[1, wh_x, \epsilon, \epsilon wh_x : x \in \mathbb{F}_p]$.

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Lemma

$\mathrm{End}(\Delta') \cong \mathbb{F}_q^{2\times 2}$ *has a Schur basis* (B_1, B_w, B_ϵ, $B_{\epsilon w} = B_\epsilon B_w$), *where* $B_\epsilon = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ *and* $B_w = \begin{pmatrix} X & Y \\ -Y^{tr} & X^{tr} \end{pmatrix}$ *with*

$$X = \begin{pmatrix} 0 & 1 & \dots & 1 \\ -1 & & & \\ \vdots & & R_X & \\ -1 & & & \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & R_Y & \\ 0 & & & \end{pmatrix}.$$

Here the rows and columns of R_X *and* R_Y *are indexed by* $\{0, \dots, p-1\}$ *the elements of* $\mathbb{F}_p$ *and*

$$(R_X)_{a,b} = \begin{cases} 0 & , b-a \notin (\mathbb{F}_p^*)^2 \\ \left(\frac{c}{p}\right) & , b-a = c^2 \in (\mathbb{F}_p^*)^2 \end{cases}, \quad (R_Y)_{a,b} = \begin{cases} 0 & , 2(b-a) \notin (\mathbb{F}_p^*)^2 \\ \left(\frac{c}{p}\right) & , 2(b-a) = c^2 \in (\mathbb{F}_p^*)^2 \end{cases}$$

The new series of Codes

Definition

Let p be a prime $p \equiv_8 5$ and assume that there is $a \in \mathbb{F}_q^*$ such that $a^2 = -tp$ for $t = 1$ or $t = 2$. We then put

$$V_t(p) := \begin{cases} aI_{2(p+1)} + B_w & , t = 1 \\ aI_{2(p+1)} + B_w + B_{\epsilon w} & , t = 2 \end{cases}$$

and let $\mathcal{V}_q(p)$ be the linear code spanned by the rows of $V_t(p)$.

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Theorem

$\mathcal{V}_q(p)$ is a self-dual code in $\mathbb{F}_q^{2(p+1)}$. Its monomial automorphism group contains the group $SL_2(p)$.

The new series of Codes

Minimum distance of $\mathcal{V}_3(p)$ computed with MAGMA:

p	5	13	29	37	53
$2(p+1)$	12	28	60	76	108
$d(\mathcal{V}_3(p))$	6	9	18	18	24
$\mathrm{Aut}(\mathcal{V}_3(p))$	$2.M_{12}$	$SL_2(13)$	$SL_2(29)$	$\geq SL_2(37)$	$\geq SL_2(53)$

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For $q = 5, 7$, and 11 and small lengths we computed $d(\mathcal{V}_q(p))$ with MAGMA:

(p, q)	$(13, 5)$	$(29, 5)$	$(5, 7)$	$(13, 7)$	$(5, 11)$	$(13, 11)$
$2(p+1)$	28	60	12	28	12	28
$d(\mathcal{V}_q(p))$	10	16	6	9	7	11

Conclusions

The matrices of rank $p+1$ in $\mathrm{End}(\Delta')$ yield $q+1$ different self-dual codes invariant under $\Delta'(\mathrm{SL}_2(p))$. In general these fall into different equivalence classes.

For instance for $q=7$, where 2 is a square mod 7, the codes spanned by the rows of $V_1(p)$ and $V_2(p)$ are inequivalent for $p=5$ and $p=13$ but have the same minimum distance.

Some related research topics are:

1. What can be established about the weight distribution of $\mathcal{V}_q(p)$ codes?
2. Are there extremal unimodular lattices related to extremal codes in the new series?
3. Do these codes yield another extremal codes?

Thanks for your attention

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