# Optimal strategies for a model of combinatorial two-sided search.

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Dedicated to the memory of Stefan Dodunekov

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# Introduction

The first developments in search theory were made by Bernard Koopman and his colleagues during World War II. The purpose was to provide efficient ways to search for enemy submarines. The work done from 1942 to 1945 was published later (1946) in a book [5].

In 1964 Berlekamp [2] showed that the problem of adaptive searching for one element with at most $e$ wrong answers is equivalent to construction of an $e$–error correcting code with feedback.

During the Workshop "Search Methodologies II" (2012) Rudolf Ahlswede suggested to consider some combinatorial models of two-sided search for a moving object. This inspired us to introduce the following two-sided search model.

# Classical group testing

- $[N] := \{1, \dots, N\}$ the set of elements
- $\mathcal{D} \subset [N]$ the set of defective elements

The classical group testing problem:
find the unknown subset $\mathcal{D}$ of all defective elements in $[N]$.

For a subset $\mathcal{S} \subset [N]$ a test $t_{\mathcal{S}}$ is the function $t_{\mathcal{S}} : 2^{[N]} \to \{0, 1\}$ with

$$t_{\mathcal{S}}(\mathcal{D}) = \begin{cases} 0 & , \text{ if } |\mathcal{S} \cap \mathcal{D}| = 0 \\ 1 & , \text{ otherwise.} \end{cases} \tag{1}$$

In classical group testing a strategy is called successful, if we can **uniquely determine** $\mathcal{D}$.

Strategies are called **adaptive** if the results of the first $k-1$ tests determine the $k$th test.
Strategies in which we choose all tests independently are called **nonadaptive**.

We consider only adaptive strategies and worst case analysis.

# A new combinatorial search problem

We define our search space $\mathcal{N} = \{1, 2, \ldots, N\}$ as the vertices of a graph $G = (\mathcal{N}, \mathcal{E})$. A searching object, also called a target, occupies one of those vertices unknown to the searcher.

Let $d_1 \in \mathcal{N}$ be the initial unknown position of the target and let $(\mathcal{T}_1, \mathcal{T}_2, \ldots, \mathcal{T}_n)$ be a sequence of test sets $\mathcal{T}_i \subset \mathcal{N}$ (tests for short) performed one after another at a time. Let also $(d_1, \ldots, d_{n+1})$ be the corresponding unknown walk performed by the target.

For each test $\mathcal{T}_i$ we define the test function

$$f_{\mathcal{T}_i}(d_i) = \begin{cases} 0 & , \text{ if } d_i \notin \mathcal{T}_i \\ 1 & , \text{ if } d_i \in \mathcal{T}_i. \end{cases}$$

# A new combinatorial search problem

We denote by $\mathcal{D}_i$ the set of possible positions of the target after the $i$th test, thus $\mathcal{D}_0 = \mathcal{N}$ and for $i = 1, \ldots, n$ we have

$$\mathcal{D}_i = \begin{cases} \Gamma(\mathcal{T}_i) & , \text{ if } f_{\mathcal{T}_i}(d_i) = 1 \\ \Gamma(\mathcal{D}_{i-1} \setminus \mathcal{T}_i) & , \text{ if } f_{\mathcal{T}_i}(d_i) = 0, \end{cases}$$

where $\Gamma(\mathcal{A}) := \{j \in \mathcal{N} : \exists i \in \mathcal{A} \text{ with } (i, j) \in \mathcal{E}\}$ is the neighborhood of a subset $\mathcal{A} \subset \mathcal{N}$.

Given a graph $G = (\mathcal{N}, \mathcal{E})$, a strategy of length $n$ is called $(G, s)$–successful if $|\mathcal{D}_i| \leq s$ for some $i \leq n$.

Let $s^*(G)$ be the minimal number $s^*$ such that there exists a $(G, s^*)$–successful strategy.

# cycles

We consider only two classes of graphs: cycles and paths on $N$ vertices. We introduce two new notation.

Given integers $n, s \geq 1$, we denote by $N_c(n, s)$ resp. $N_l(n, s)$ the maximal number $N$, such that there exists a $(C_N, s)$–successful resp. $(C_L, s)$–successful strategy.

We start with a simple observation for cycles.

**Proposition**

*For $N \geq 5$ we have $s^*(C_N) = 5$.*

**Theorem**

*For $n \geq 0$ we have*

$$N_c(n, 5) = 2^n + 4.$$

Next we consider a path as an underlying graph.

**Proposition**

*For $N \geq 4$ we have $s^*(C_L) = 4$.*

**Theorem**

*For integers $n \geq 0$ and $s \geq 4$ we have*

$$N_l(n,s) = (s-4)2^n + 2n + 4.$$

# A restricted case of the problem, when the number of moves of the target is limited

In this section we consider the case when the target can move at most $t$ times.

We describe now a coding problem which is equivalent to our two-sided search model. Let $\mathcal{N} = \{1, 2, \ldots, N\}$ be a set of messages, which we identify with the vertices of an undirected graph $G = (\mathcal{N}, \mathcal{E})$. A source chooses a message $d_1 \in \mathcal{N}$ which the transmitter should transmit by sending at most $n(G)$ binary symbols (bits) step by step (adaptively) over a noiseless binary channel. However, after every transmission of one bit, the source may change the message into a neighboring message. The sequence of vertices $d_1, \ldots, d_j$ describes an alteration, after $j$ transmissions with the actual message $d_j$. Let $(c_1, \ldots, c_{j-1}) \in \{0, 1\}^{j-1}$ be the submitted sequence of the sender. Then the $j$th bit $c_j$ depends on the actual message $d_j$ and the $j - 1$ submitted bits, so that $c_j = c_j(c_1, \ldots, c_{j-1}, d_j)$.

# A restricted case of the problem, when the number of moves of the target is limited

The goal is to describe an efficient scheme of transmission such that for every walk $d_1, \ldots, d_{n+1}$ the receiver is able to find a set $S \subset \mathcal{N}$, of a given size $s$, which includes a message $d_{j+1}$, after $j \leq n$ transmissions. It can be seen that this setting of the problem is equivalent to our search problem. On the other hand we note that from the coding point of view it seems more natural to consider the following problem: the goal is to find a set of size $s$ containing the message $d_{n+1}$.

We emphasize that for cycles and paths the answers for both problems are the same.

# A restricted case, cycles

**Theorem**

*For integers $s \geq 5$, $1 \leq t < n$ we have*

$$N_c(n, s, t) \geq (s-4)2^n + 4 + 4(2^{n-t} - 1).$$

We consider now the cases $t = 1, 2$.

**Theorem**

(i) *For $n \geq 1$ and $s \geq 3$ we have $N_c(n, s, 1) = (s-2)2^n$.*

(ii) *For $n \geq 4$ we have $N_c(n, 3, 2) = 2^{n-2}$.*

# A restricted case, paths

Next we consider the restricted case for paths.

**Theorem**

*For integers $s \geq 5$, $1 \leq t < n$ we have*

$$N_l(n, s, t) \geq (s-4)2^n + 2t + 2^{n-t+2}.$$

**Theorem**

*For $s \geq 3$ we have*

$$N_l(n, s, 1) = (s-2)2^n + 2.$$

We have considered a two-sided combinatorial search problem for two classes of underlying graphs, cycles and paths, with the most simple topologies.

In fact, the problem essentially depends on the topology of the underlying graph.

It is natural to consider the problem for other popular topologies like grids, trees, $n$-cubes etc.

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Thank you for your attention!