MDS deformations of linear codes

Azniv Kasparian, Evgeniya Velikova

1. Existence of MDS deformations
2. The MDS deformations as $\mathbb{F}_q$-Zariski tangent bundles

Convention

- Throughout, an $[n, k, d]_q$-code is an $\mathbb{F}_q$-linear subspace $C \subset \mathbb{F}_q^n$ of $\dim_{\mathbb{F}_q} C = k$, such that any $c \in C \setminus \{(0, \dots 0)\}$ has at least d nonzero components.
- Singleton bound $k + d \leq n + 1$ is attained by the $[n, k, n-k+1]_q$-codes, which are called MDS (Maximum Distance Separable).

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- Singleton bound $k + d \leq n + 1$ is attained by the $[n, k, n - k + 1]_q$-codes, which are called MDS (Maximum Distance Separable).

Jacobian matrix of polynomials

- Let $\mathbb{F}_q$ be a finite field of $\mathrm{char}(\mathbb{F}_q) = p$, $1 \leq j \leq n$, $\frac{\partial}{\partial x_j} : \mathbb{F}_q[x_1, \ldots, x_n] \longrightarrow \mathbb{F}_q[x_1, \ldots, x_n]$, $\frac{\partial}{\partial x_j} \left(\sum_{\alpha} c_\alpha x_1^{\alpha_1} \ldots x_j^{\alpha_j} \ldots x_n^{\alpha_n} \right) =$ $\sum_{\alpha} c_\alpha x_1^{\alpha_1} \ldots [\alpha_j (\mathrm{mod}\, p)] x_j^{\max(0, \alpha_j - 1)} \ldots x_n^{\alpha_n}$.
- Consider the Jacobian matrix

$$\frac{\partial(f_1, \ldots, f_{n-k})}{\partial(x_1, \ldots, x_n)}(a) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \ldots & \frac{\partial f_1}{\partial x_n}(a) \\ \ldots & \ldots & \ldots \\ \frac{\partial f_{n-k}}{\partial x_1}(a) & \ldots & \frac{\partial f_{n-k}}{\partial x_n}(a) \end{pmatrix}$$

of $f_1, \ldots, f_{n-k} \in \mathbb{F}_q[x_1, \ldots, x_n]$ at $a \in \mathbb{F}_q^n$.

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Jacobian family of linear codes

- The union $J(f_1, \ldots, f_{n-k}) = \cup_{a \in \mathbb{F}_q^n} J(f_1, \ldots, f_{n-k})_a$ of the codes with check matrices $\frac{\partial(f_1,\ldots,f_{n-k})}{\partial(x_1,\ldots,x_n)}(a)$ is referred to as the Jacobian family of linear codes, associated with $f_1, \ldots, f_{n-k} \in \mathbb{F}_q[x_1, \ldots, x_n]$.

- Thus, $J(f_1, \ldots, f_{n-k})_a$ are $\mathbb{F}_q$-linear codes of length n and dimension $n - \mathrm{rk}\frac{\partial(f_1,\ldots,f_{n-k})}{\partial(x_1,\ldots,x_n)}(a) \geq k$.

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Existence of an MDS deformation

- For an arbitrary code $C_0 \subset \mathbb{F}_q^n$ and arbitrary $[n, k, n-k+1]_q$-codes $C_1, \ldots, C_r$, $r \leq q-1$ there exists a Jacobian family $J(f_1, \ldots, f_{n-k}) \to \mathbb{F}_q^n$ with $J(f_1, \ldots, f_{n-k})_{a^{(i)}} = C_i$ for some $a^{(0)}, \ldots, a^{(r)} \in \mathbb{F}_q^n$.
- Let $\mathbb{F}_q = \{t_0 = 0, t_1, \ldots, t_{q-1}\}$, $p = \mathrm{char}(\mathbb{F}_q)$ and $\Phi_p : \mathbb{F}_q \to \mathbb{F}_q$, $\Phi_p(t) = t^p$ be the Frobenius automorphism.

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Lagrange interpolation step

- If $A^{(i)} = (A_1^{(i)} \dots A_n^{(i)})$ are check matrices of C_i and
- $L_i(x) = \frac{(x-t_0)\dots(x-t_{i-1})(x-t_{i+1})\dots(x-t_r)}{(t_i-t_0)\dots(t_i-t_{i-1})(t_i-t_{i+1})\dots(t_i-t_r)}$ are the Lagrange basis polynomials for $r+1$ points then
- $H_j(x_j) = \sum_{i=0}^{r} A_j^{(i)} L_i(x_j^p) \in Mat_{(n-k)\times 1}(\mathbb{F}_q[x_j])$ pass through $H_j(\Phi_p^{-1}(t_i)) = A_j^{(i)}$ for $\forall 0 \le i \le r \le q-1$.

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"Integration" step

Due to $\frac{\partial(x_j L_i(x_j^p))}{\partial x_j} = L_i(x_j^p)$, the polynomials

$$f_s(x_1, \ldots, x_n) = \sum_{j=1}^{n} \sum_{i=0}^{r} A_{sj}^{(i)} x_j L_i(x_j^p), \quad 1 \leq s \leq n-k$$

have Jacobian matrix $\frac{\partial(f_1,\ldots,f_{n-k})}{\partial(x_1,\ldots,x_n)} = (H_1(x_1) \ldots H_n(x_n))$.

Preparation for arc-interpretation

- Consider the space $\mathbb{F}_q[x_1, \ldots, x_n]^{n-k} \ni (f_1, \ldots, f_{n-k})$ of the morphisms $(f_1, \ldots, f_{n-k}) : \mathbb{F}_q^n \to \mathbb{F}_q^{n-k}$ and the space $\mathbb{F}_q[x_1, \ldots, x_n]^{n-k}/\mathbb{F}_q^* \ni [f_1 : \ldots : f_{n-k}]$ of the rational maps $[f_1 : \ldots : f_{n-k}] : \mathbb{F}_q^n \dashrightarrow \mathbb{P}^{n-k-1}(\mathbb{F}_q)$.

- The derivations $\frac{\partial}{\partial x_j}$, $1 \leq j \leq n$ commute with the $\mathbb{F}_q^*$-action $\mathbb{F}_q^* \times \mathbb{F}_q[x_1, \ldots, x_n]^{n-k} \longrightarrow \mathbb{F}_q[x_1, \ldots, x_n]^{n-k}$, $(\lambda, (f_1, \ldots, f_{n-k})) \mapsto (\lambda f_1, \ldots, \lambda f_{n-k})$ and descend to $\frac{\partial}{\partial x_j} : \mathbb{F}_q[x_1, \ldots, x_n]^{n-k}/\mathbb{F}_q^* \longrightarrow \mathbb{F}_q[x_1, \ldots, x_n]^{n-k}/\mathbb{F}_q^*$.

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The counterpart for arcs

- Suppose that the points of the arcs $\mathcal{A}_i = \{P_1^{(i)}, \ldots, P_n^{(i)}\} \subset \mathbb{P}^{n-k-1}(\mathbb{F}_q)$, $1 \leq i \leq q-1$ lift to $\widetilde{P_j^{(i)}} \in \mathbb{F}_q^{n-k}$ with $\widetilde{P_1^{(i)}} + \ldots + \widetilde{P_n^{(i)}} \neq (0, \ldots, 0)$.
- Then there is a rational map $[f_1 : \ldots : f_{n-k}] : \mathbb{F}_q^n \dashrightarrow \mathbb{P}^{n-k-1}(\mathbb{F}_q)$, such that all $\frac{\partial}{\partial x_j}[f_1 : \ldots : f_{n-k}] : \mathbb{F}_q^n \dashrightarrow \mathbb{P}^{n-k-1}(\mathbb{F}_q)$ pass through $\frac{\partial}{\partial x_j}[f_1 : \ldots : f_{n-k}](\Phi_p^{-1}(t_i), \ldots, \Phi_p^{-1}(t_i)) = P_j^{(i)}$.

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The rational normal curve

If $\mathbb{F}_q = \{t_0 = 0, t_1, \dots, t_{q-1}\}$ then the rational normal curve

$$\mathcal{A}_\sigma = \left\{ \left[1 : t_i : \dots : t_i^{q-k} \right] \,\middle|\, 0 \leq i \leq q-1 \right\} \cup \{\infty = [0 : \dots : 0 : 1]\}$$

is a $(q+1)$-arc in $\mathbb{P}^{q-k}(\mathbb{F}_q)$.

A family with q! MDS-fibers

- If $S_d = x_1^d + \ldots + x_q^d$, $p = \mathrm{char}(\mathbb{F}_q)$ and $a = (a', a_{q+1}) \in \mathbb{F}_q^{q+1}$ is a point with different $a_1, \ldots, a_q \in \mathbb{F}_q$ then the fiber J_a of $J = J(S_1, S_{p+1}, \ldots, S_{(q-k-1)p+1}, S_{(q-k)p+1} + x_{q+1})$ is associated with $\mathcal{A}$ and, therefore, is an $[q+1, k, q-k+2]_q$-code.
- If $a' = (a_1, \ldots, a_q) \in \mathbb{F}_q^q$ has $q+1-k \leq t \leq q-1$ different coordinates then $J_{(a', a_n)}$ is an $[q+1, k, 2]_q$-code.
- When $a' \in \mathbb{F}_q^q$ has $1 \leq t \leq q-k$ different components, $J_{(a', a_n)}$ is a $[q+1, q+1-t, 2]_q$-code with $q+1-t > k$.

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Affine variety, defined over $\mathbb{F}_q$

- Let $\overline{\mathbb{F}_q} = \cup_{s=1}^{\infty} \mathbb{F}_{q^s}$ be the algebraic closure of $\mathbb{F}_q$.
- For any $g_1, \ldots, g_m \in \mathbb{F}_q[x_1, \ldots, x_n]$ the set $X = V(g_1, \ldots, g_m) = \{a \in \overline{\mathbb{F}_q}^n \mid g_i(a_1, \ldots, a_n) = 0, \ \forall 1 \leq i \leq m\}$ is an affine variety, defined over $\mathbb{F}_q$ and
- $X(\mathbb{F}_q) := X \cap \mathbb{F}_q^n$ is the set of the $\mathbb{F}_q$-rational points of X.

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Zariski topology

- The collection $\mathcal{C}$ of $V(g_1, \ldots, g_m) \subseteq \overline{\mathbb{F}_q}^n$ for all $g_1, \ldots, g_m \in \overline{\mathbb{F}_q}[x_1, \ldots, x_n]$ is a family of closed subsets since
- $\emptyset, \overline{\mathbb{F}_q}^n \in \mathcal{C}$,
- if $\forall X_i \in \mathcal{C}$ then $X_1 \cup \ldots \cup X_m \in \mathcal{C}$ and
- if $\forall X_\alpha \in \mathcal{C}$ then $\cap_{\alpha \in A} X_\alpha \in \mathcal{C}$.
- The Zariski topology on $\overline{\mathbb{F}_q}^n$ has closed sets $\mathcal{C}$.

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Geometric characterization of the dimension

- A morphism $\varphi = (\varphi_1, \ldots, \varphi_s) : X \to \overline{\mathbb{F}_q}^s$, given by polynomials $\varphi_1, \ldots, \varphi_s \in \overline{\mathbb{F}_q}[x_1, \ldots, x_n]$ is dominant if its image $\varphi(X)$ is Zariski dense in $\overline{\mathbb{F}_q}^s$.

- A morphism $\varphi = (\varphi_1, \ldots, \varphi_s) : X \to \overline{\mathbb{F}_q}^s$ is finite if there is a non-empty, Zariski open, Zariski dense subset $U \subseteq \overline{\mathbb{F}_q}^s$, such that the fibers of $\varphi : \varphi^{-1}(U) \to U$ are finite sets.

- The dimension of an affine variety $X \subset \overline{\mathbb{F}_q}^n$ is the natural number k, for which there exists a finite dominant morphism $\varphi : X \to \overline{\mathbb{F}_q}^k$.

Geometric characterization of the dimension

- A morphism $\varphi = (\varphi_1, \dots, \varphi_s) : X \to \overline{\mathbb{F}_q}^s$, given by polynomials $\varphi_1, \dots, \varphi_s \in \overline{\mathbb{F}_q}[x_1, \dots, x_n]$ is dominant if its image $\varphi(X)$ is Zariski dense in $\overline{\mathbb{F}_q}^s$.
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The Zariski and the $\mathbb{F}_q$-Zariski tangent bundles

- Let $I = \langle g_1, \ldots, g_m \rangle \lhd \mathbb{F}_q[x_1, \ldots, x_n]$, $X = V(I) \subseteq \overline{\mathbb{F}_q}^n$, $I(X) = \{h \in \mathbb{F}_q[x_1, \ldots, x_n] \,|\, h(a) = 0, \forall a \in X\} = \langle h_1, \ldots, h_s \rangle \lhd \mathbb{F}_q[x_1, \ldots, x_n]$ be the ideal of X over $\mathbb{F}_q$, $\overline{I}(X) = \{h \in \overline{\mathbb{F}_q}[x_1, \ldots, x_n] \,|\, h(a) = 0, \forall a \in X\} = \langle h_1, \ldots, h_s, \ldots h_r \rangle \lhd \overline{\mathbb{F}_q}[x_1, \ldots, x_n]$ be the ideal of X over $\overline{\mathbb{F}_q}$.
- The $\mathbb{F}_q$-Zariski tangent bundle to X is
$$T^{\mathbb{F}_q}X := J(h_1, \ldots, h_s)|_{X(\mathbb{F}_q)}.$$
- The Zariski tangent bundle to X is
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Relating Zariski and $\mathbb{F}_q$-Zariski tangent spaces

- $I = \langle g_1, \ldots, g_m \rangle \subseteq I(X)$ and $I \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \subseteq I(X) \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \subseteq \bar{I}(X)$ imply

$$T_a^{\mathbb{F}_q} X \subseteq J(g_1, \ldots, g_m)_a \quad \text{and}$$

$$T_a X \subseteq T_a^{\mathbb{F}_q} X \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \subseteq J(g_1, \ldots, g_m)_a \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q} \quad \text{for} \quad \forall a \in X(\mathbb{F}_q).$$

- Problem: Find a sufficient condition on $g_1, \ldots, g_m$, such that

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Smooth and singular points

- If $\dim_{\overline{\mathbb{F}_q}} T_a X = \dim X$ then $a \in X$ is a smooth point.
- The smooth locus X^{smooth} of X is non-empty, Zariski open, Zariski dense subset of X.
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Realization of $J(f_1, \dots, f_{n-k})$ as $T^{\mathbb{F}_q}X$

- Let $J(f_1, \dots, f_{n-k}) \to \mathbb{F}_q^n$ be a Jacobian family with $\dim_{\mathbb{F}_q} J(f_1, \dots, f_{n-k})_a = k$ for $\forall a \in S_o \subseteq \mathbb{F}_q^n$, $D = \max(\deg(f_1), \dots, \deg(f_{n-k}))$, $p = \mathrm{char}(\mathbb{F}_q)$.
- Then the polynomials $g_s(x_1, \dots, x_n) = f_s(x_1, \dots, x_n) + x_s^{pD}$, $1 \leq s \leq n-k$ provide an affine variety $X = V(g_1, \dots, g_{n-k}) \subset \overline{\mathbb{F}_q}^n$ of $\dim X = k$, such that
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- If $J(f_1, \ldots, f_{n-k})_{a^{(\lambda)}}$ are $[n, k, n-k+1]_q$-codes for $1 \leq \lambda \leq q$,
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Global geometric characterization of the maximum distance separability of an $\mathbb{F}_q$-Zariski tangent space

- Let $X = V(g_1, \ldots, g_{n-k}) \subset \overline{\mathbb{F}_q}^n$, $g_s \in \mathbb{F}_q[x_1, \ldots, x_n]$ be an affine variety of $\dim X = k$. For $\forall i = (i_1, \ldots, i_k)$ consider the projection $\Pi_i : X \rightarrow \overline{\mathbb{F}_q}^k$, $\Pi_i(x_1, \ldots, x_n) = (x_{i_1}, \ldots, x_{i_k})$.
- If $T_a^{\mathbb{F}_q} X = J(g_1, \ldots, g_{n-k})_a$, $a \in X^{smooth}(\mathbb{F}_q)$ is an $[n, k, n-k+1]_q$-code then the projections Π_i are dominant for all $i = (i_1, \ldots, i_k)$, $1 \leq i_1 < \ldots < i_k \leq n$.
- If Π_i are dominant for all $i = (i_1, \ldots, i_k)$ then there exists $N \in \mathbb{N}$, depending on X and on the embedding of X in $\overline{\mathbb{F}_q}^n$, such that for any $m \in \mathbb{N}$ with $q^m > N$ the spaces $T_a^{\mathbb{F}_{q^m}} X = J(g_1, \ldots, g_{n-k})_a$, $a \in X^{smooth}(\mathbb{F}_{q^m})$ are $[n, k, n-k+1]_{q^m}$-codes.

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Global geometric characterization of the maximum distance separability of an $\mathbb{F}_q$-Zariski tangent space

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The ideal of the leading terms

- Let us endow the monomials of $x_1, \ldots, x_n$ by the lexicographic order $x_1^{\lambda_1} \ldots x_n^{\lambda_n} \succ x_1^{\mu_1} \ldots x_n^{\mu_n}$ if and only if there is $1 \leq j \leq n$ with $\lambda_1 = \mu_1, \ldots, \lambda_{j-1} = \mu_{j-1}$, $\lambda_j > \mu_j$.
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Groebner basis

- If $LT(I) = \langle LT(\gamma_1), \ldots, LT(\gamma_s) \rangle \triangleleft \mathbb{F}_q[x_1, \ldots, x_n]$ for some polynomials $\gamma_1, \ldots, \gamma_s \in I$ then $I = \langle \gamma_1, \ldots, \gamma_s \rangle$ and $G = \{\gamma_1, \ldots, \gamma_s\}$ is a Groebner basis of I with respect to the considered monomial order.
- $\Pi_{(n-k+1,\ldots,n)} : V(I) \rightarrow \overline{\mathbb{F}_q}^k$, $\Pi(x_1, \ldots, x_n) = (x_{n-k+1}, \ldots, x_n)$ is dominant if and only if $G \cap \mathbb{F}_q[x_{n-k+1}, \ldots, x_n] = \emptyset$ for any Groebner basis G of I with respect to the lexicographic order.

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The designed minimum distance of a tangent space

- Let $X = V(g_1, \dots, g_{n-k}) \subset \overline{\mathbb{F}_q}^n$, $g_s \in \mathbb{F}_q[x_1, \dots, x_n]$ be an affine variety of $\dim X = k$. For $\forall \mu = (\mu_1, \dots, \mu_{n-d})$ consider $\Pi_\mu : X \to \overline{\mathbb{F}_q}^{n-d}$, $\Pi_\mu(x_1, \dots, x_n) = (x_{\mu_1}, \dots, x_{\mu_{n-d}})$.
- If $T_a^{\mathbb{F}_q} X = J(g_1, \dots, g_{n-k})_a$, $a \in X^{smooth}(\mathbb{F}_q)$ is an $[n, k, d_o]_q$-code with $d_o > d$ then $\dim \Pi_\mu(X) = k$ for $\forall \mu$.
- If $\dim \Pi_\mu(X) = k$ for $\forall \mu$ then there exists $N \in \mathbb{N}$, such that for any $m \in \mathbb{N}$ with $q^m > N$ the spaces $T_a^{\mathbb{F}_{q^m}} X = J(g_1, \dots, g_{n-k})_a$, $a \in X^{smooth}(\mathbb{F}_{q^m})$ are $[n, k, d_o]_{q^m}$-codes with $d_o > d$.

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Thank you very much for your attention!