# A NOTE ON THE EXISTENCE OF SPREADS IN PROJECTIVE HJELMSLEV SPACES

Nevyana Georgieva

Ivan Landjev

New Bulgarian University

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# 1. Modules over finite chain rings

**Theorem.** Let $R$ be a finite chain ring of nilpotency index $m$. For any finite module ${}_RM$ there exists a uniquely determined partition

$$\lambda = (\lambda_1, \ldots, \lambda_k) \vdash \log_q |M|,$$

$0 \leq \lambda_i \leq m$, such that

$${}_RM \cong R/(\operatorname{rad} R)^{\lambda_1} \oplus \ldots \oplus R/(\operatorname{rad} R)^{\lambda_k}.$$

The partition $\lambda$ is called the **shape** of ${}_RM$.

The number $k$ is called the **rank** of ${}_RM$.

# 2. Projective Hjelmslev spaces

- $M = {}_R R^k$; $M^* := M \setminus \theta M$; $\theta \in \operatorname{rad} R \setminus (\operatorname{rad} R)^2$
- $\mathcal{P} = \{Rx \mid x \in M^*\}$;
- $\mathcal{L} = \{Rx + Ry \mid x, y$ linearly independent$\}$;
- $I \subseteq \mathcal{P} \times \mathcal{L}$ – incidence relation;
- $\frown\!\!\!\!\!\smile$ - **neighbour relation**:

  (N1) $X \frown\!\!\!\!\!\smile Y$ if $\exists s, t \in \mathcal{L}$: $X, Y\, I\, s, X, Y\, I\, t$;

  (N2) $s \frown\!\!\!\!\!\smile t$ if $\forall X\ I\ s\ \exists Y\ I\ t$: $X \frown\!\!\!\!\!\smile Y$ and $\forall Y\ I\ t\ \exists X\ I\ s$: $Y \frown\!\!\!\!\!\smile X$.

**Definition.** The incidence structure $\Pi = (\mathcal{P}, \mathcal{L}, I)$ with neighbour relation $\frown\!\!\!\!\!\smile$ is called the (**left**) **projective Hjelmslev geometry** over the chain ring $R$.

**Definition.** A set of points $H$ in the projective Hjelmslev space $\Pi$ is called a **Hjelmlsev subspace** if for any two points $x, y \in H$ there is at least one line incident with both of them which is entirely contained in $H$.

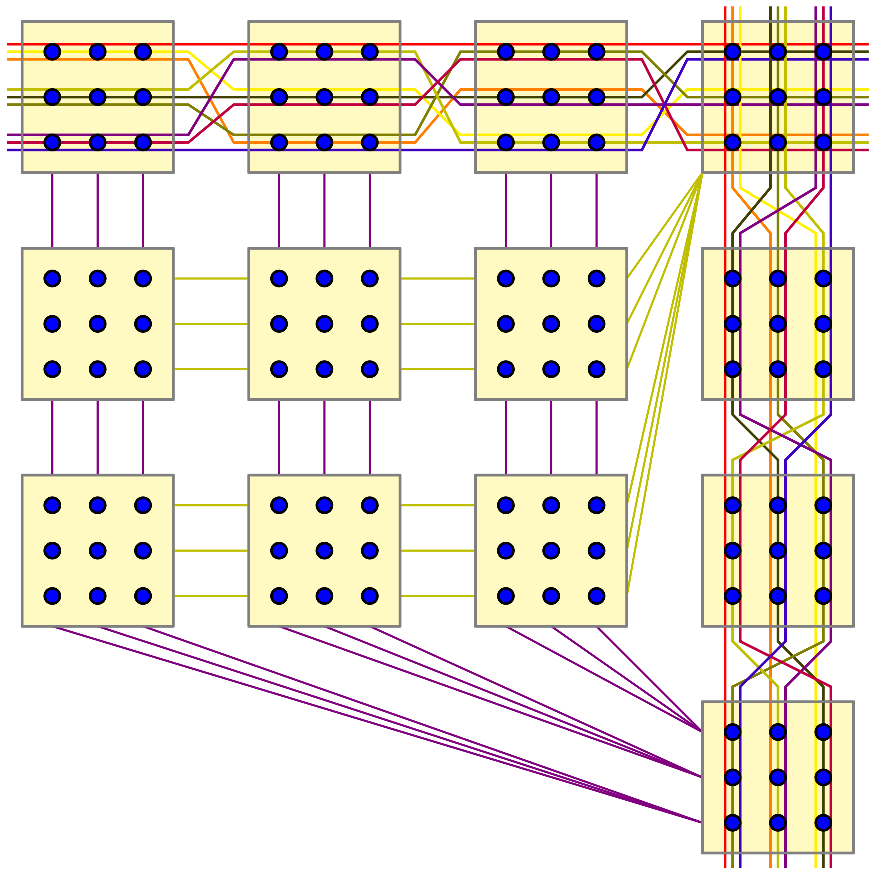
**Definition.** A set of points $H$ in the projective Hjelmlsev space $\Pi$ is called a **subspace** if it is the intersection of Hjelmlsev subspaces.

Hjelmslev subspaces $\longrightarrow$ free submodules of ${}_RR^n$

subspaces $\longrightarrow$ submodules of ${}_RR^n$ with at least one free submodule

subspace of type $\lambda$ $\longrightarrow$ submodule of type $\lambda$

# $\mathrm{PHG}(\mathbb{Z}_9^3)$



# 3. The Lattice of Submodules

**Theorem.** Let ${}_RM$ be a module of shape $\lambda = (\lambda_1, \ldots, \lambda_n)$. For every sequence $\mu = (\mu_1, \ldots, \mu_n)$, $\mu_1 \geq \ldots \geq \mu_n \geq 0$, satisfying $\mu \leq \lambda$ the module ${}_RM$ has exactly

$$\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{q^m} = \prod_{i=1}^{m} q^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \cdot \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \mu'_i - \mu'_{i+1} \end{bmatrix}_q$$

submodules of shape $\mu$. In particular, the number of free rank $s$ submodules of ${}_RM$ equals

$$q^{s(\lambda'_1 - s) + \ldots + s(\lambda'_{m-1} - s)} \cdot \begin{bmatrix} \lambda'_m \\ s \end{bmatrix}_q .$$

Here

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q^n - 1) \ldots (q^{n-k+1} - 1)}{(q^k - 1) \ldots (q - 1)}.$$

are the Gaussian coefficients.

**Example.** Let

- $\mathbb{Z}_4^4 = \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4$
- $q^m = 2^2$, i.e. $q = 2$, $m = 2$
- $\lambda = (2, 2, 2, 2)$, $\lambda' = (4, 4)$;
- $\mu = (2, 2, 1, 0)$, $\mu' = (3, 2)$
- $\begin{bmatrix} \lambda \\ \mu \end{bmatrix}_{2^2} = 2^{2(4-3)} \begin{bmatrix} 4-2 \\ 3-2 \end{bmatrix}_2 \begin{bmatrix} 4-0 \\ 2-0 \end{bmatrix}_2 = 2^2 \cdot 3 \cdot 35 = 420$.

**Theorem.** Let

$$\boldsymbol{m} = (\underbrace{m, \ldots, m}_{n}) \quad \text{and } \mu = (\mu_1, \ldots, \mu_n).$$

Set $\overline{\mu} = (m - \mu_n, \ldots, m - \mu_1)$. Then

$$\begin{bmatrix} \boldsymbol{m} \\ \mu \end{bmatrix}_{q^m} = \begin{bmatrix} \boldsymbol{m} \\ \overline{\mu} \end{bmatrix}_{q^m}.$$

Let $R$ be a chain ring with $|R| = q^m$, $R/\operatorname{rad} R \cong \mathbb{F}_q$.

Let $\kappa = (\kappa_1, \ldots, \kappa_n)$, $m \geq \kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n \geq 0$.

$\mathcal{G}_R(n, \kappa)$ – the set of all submodules of ${}_R R^n$ of shape $\kappa$.

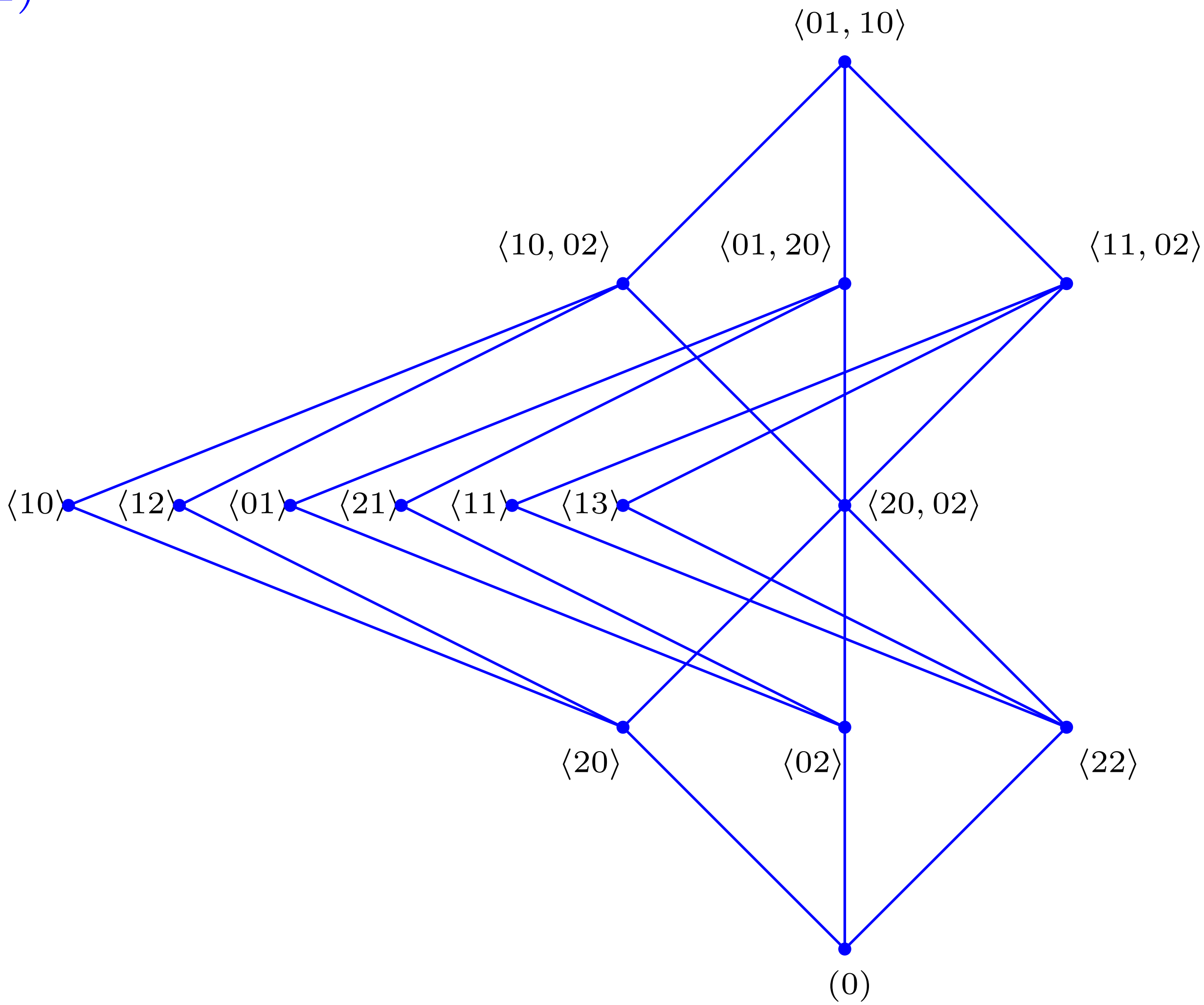
$\mathcal{H}_R(\kappa)$ – the lattice of all submodules of

$$R/(\operatorname{rad} R)^{\kappa_1} \oplus \ldots \oplus R/(\operatorname{rad} R)^{\kappa_n},$$

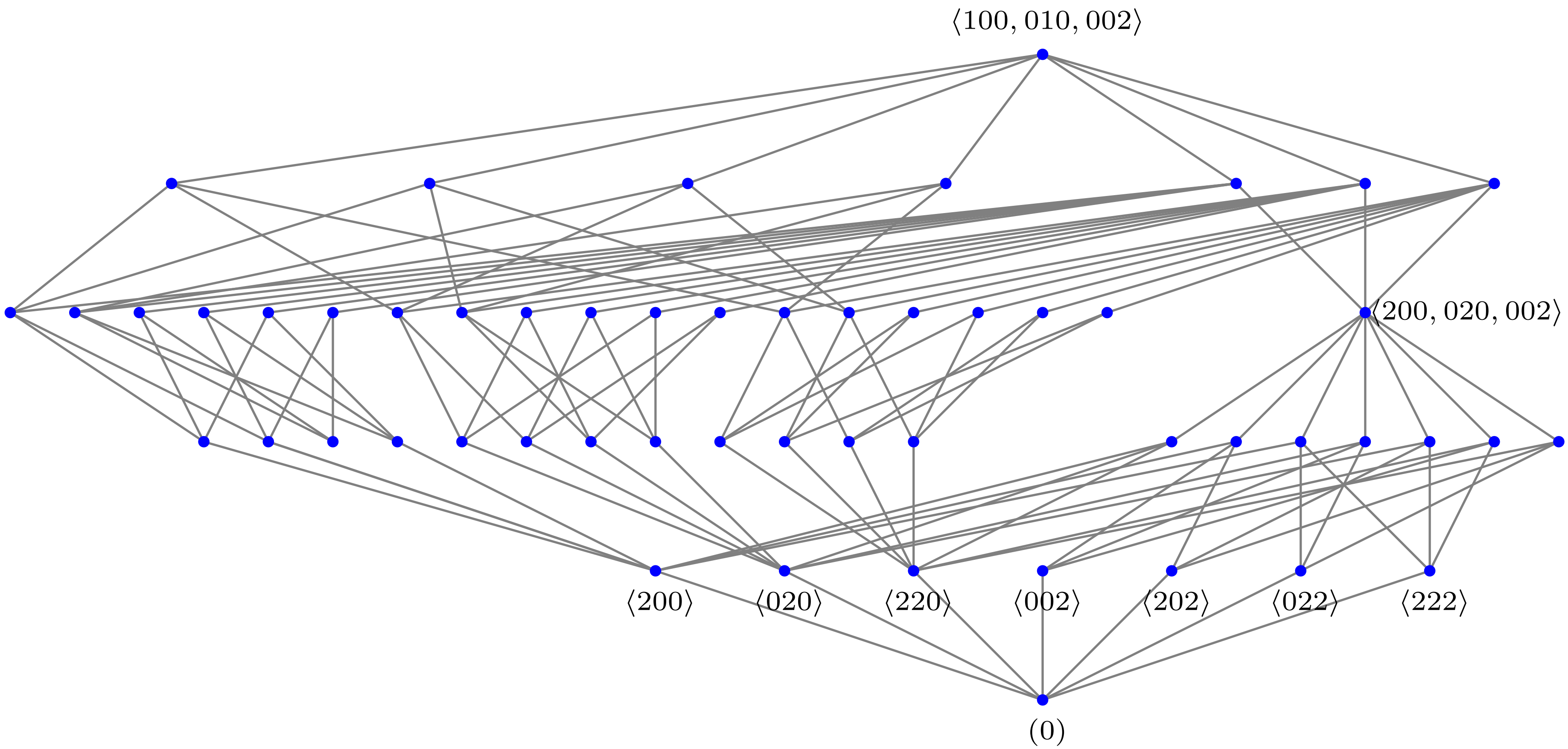
ordered by inclusion.

$\mathcal{H}_R(n)$ – the lattice of all submodules of ${}_R R^n$.

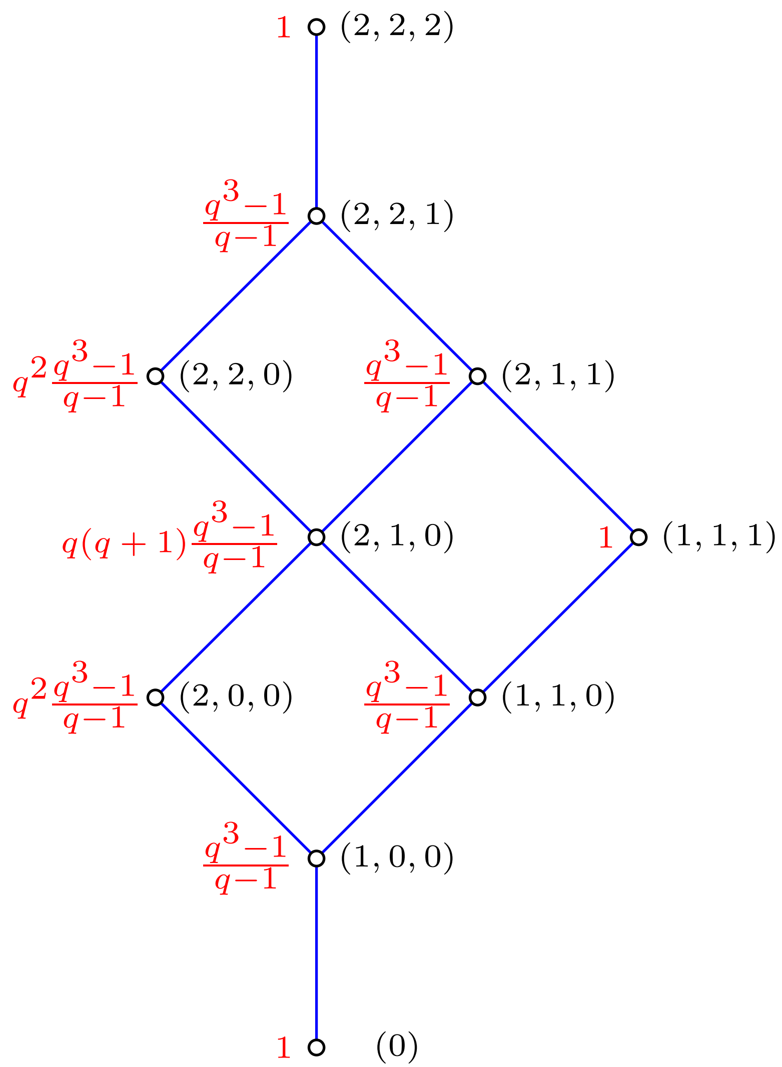
$\mathcal{H}_{\mathbb{Z}_4}(2)$



## $\mathcal{H}_R(\kappa)$, $\kappa = (2, 2, 1)$



## $\mathcal{H}_{\mathbb{Z}_4}(3)$



## 3. Spreads

**Definition.** A $r$-**spread** of the projective Hjelmslev geometry $\mathrm{PHG}({}_RR^{n+1})$ is a set $\mathcal{S}$ of $r$-dimensional Hjelmslev subspaces such that every point is contained in exactly one subspace of $\mathcal{S}$.

**Theorem.** Let $R$ be a chain ring with $|R| = q^2$, $R/\operatorname{rad} R \cong \mathbb{F}_q$. There exists a spread $\mathcal{S}$ of $r$-dimensional spaces of the $n$-dimensional projective Hjelmslev geometry $\mathrm{PHG}({}_RR^{n+1})$ if and only if $r+1$ divides $n+1$.

**Theorem.** Let $R$ be a chain ring with $|R| = q^m$, $R/\operatorname{rad} R \cong \mathbb{F}_q$. There exists a spread $\mathcal{S}$ of $r$-dimensional Hjelmslev subspaces of $\mathrm{PHG}({}_RR^n)$ if and only if $r+1$ divides $n+1$.

The factor-image of the spreads from the previous constructions is $q^{n-r}\overline{\mathcal{S}}$.

Such spreads can be considered as trivial. Do there exist non-trivial sprads?

Very interesting: construct spreads in which no two subspaces are neighbours.

In case of $\mathrm{PHG}({}_R R^4)$ such spreads do exist for

$R = \mathbb{Z}_4$, $\mathbb{F}_2[X]/(X^2)$, $\mathbb{Z}_9$, $\mathbb{F}_3[X]/(X^2)$.

(a computational result)

$\mathrm{PHG}(\mathbb{Z}_4^4)$:

| | | | |
|---|---|---|---|
| $\langle 1001, 0121\rangle$ | $\langle 2103, 0011\rangle$ | $\langle 1020, 0121\rangle$ | $\langle 0010, 2201\rangle$ |
| $\langle 0103, 2010\rangle$ | $\langle 1023, 0113\rangle$ | $\langle 1002, 0210\rangle$ | $\langle 1000, 0100\rangle$ |
| $\langle 1003, 0110\rangle$ | $\langle 1010, 0021\rangle$ | $\langle 1302, 0212\rangle$ | $\langle 1330, 0201\rangle$ |
| $\langle 1030, 0122\rangle$ | $\langle 1102, 0211\rangle$ | $\langle 0130, 0001\rangle$ | $\langle 1011, 0112\rangle$ |
| $\langle 1202, 0013\rangle$ | $\langle 1032, 0111\rangle$ | $\langle 1021, 0120\rangle$ | $\langle 1013, 0102\rangle$ |

$\kappa = (\kappa_1, \ldots, \kappa_n)$

**Definition.** A $\kappa$-**spread** of the projective Hjelmslev geometry $\mathrm{PHG}(R_R^n)$ is a set $\mathcal{S}$ of subspaces of type $\kappa$ such that every point is contained in exactly one subspace of $\mathcal{S}$.

$\kappa$-spreads are exactly the $\tau - (n, \kappa, 1)$-designs with $\tau = (m, 0, \ldots, 0)$.

Take $\kappa = (\underbrace{2, \ldots, 2}_{n/2}, \underbrace{1, \ldots, 1}_{n/2-1}, 0)$.

The number of points in a subspace of type $\kappa$ is $q^{n-2}\frac{q^{\frac{n}{2}}-1}{q-1}$ and divides the number of points in $\mathrm{PHG}\,{}_R R^n$ which is $q^{n-1}\frac{q^n-1}{q-1}$.

**Theorem.** Let $R$ be a chain ring of nilpotency index 2. Let $\Pi = \mathrm{PHG}({}_R R^n)$. There exists no $\kappa$-spread of $\Pi$ for $\kappa = (\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2-1},0)$.

**Corollary.** There exists no $\kappa$-spread of $\mathrm{PHG}({}_R R^4)$ with $\kappa = (2,2,1,0)$.

More generally:

**Theorem.** Let $R$ be a chain ring of nilpotency index m. Let $\Pi = \mathrm{PHG}({}_R R^n)$. There exists no $\kappa$-spread of $\Pi$ for $\kappa = (\underbrace{m, \ldots, m}_{n/2}, \underbrace{m-1, \ldots, m-1}_{n/2-1}, 0)$.

**Problem.** Find a necessary and sufficient condition on $\kappa$ for the existence $\kappa$-spread in $\mathrm{PHG}({}_R R^n)$.

**Theorem.** Let $R$ be a finite chain ring of nilpotency index 2 and let $\Pi = \mathrm{PHG}({}_R R^n)$ be the corresponding (left) projective Hjelmslev space. There exists no $\lambda$-spread of $\Pi = \mathrm{PHG}({}_R R^n)$ with $\lambda = (\underbrace{2, \ldots, 2}_{n/2}, \underbrace{1, \ldots, 1}_{n/2-a}, \underbrace{0, \ldots, 0}_{a})$, where

$1 \leq a \leq \frac{n}{2} - 1$.

| Shape | Existence |
| --- | --- |
| $(\underbrace{2,\ldots,2}_{n/2},\underbrace{0,\ldots,0}_{n/2})$ | YES |
| $(\underbrace{2,\ldots,2}_{n/2},1,\underbrace{0,\ldots,0}_{n/2-1})$ | NO |
| $(\underbrace{2,\ldots,2}_{n/2},1,1\,\underbrace{0,\ldots,0}_{n/2-2})$ | NO |
| . . . | . . . |
| $(\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2-1},0)$ | NO |
| $(\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2})$ | YES |