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Nonexistence of certain binary orthogonal arrays

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Orthogonal arrays

$H(n,2)$ - the binary Hamming space of dimension n.

Binary orthogonal array (BOA) in $H(n,2)$ – an $M \times n$ matrix such that every $M \times \tau$ submatrix contains as rows all ordered τ-tuples of $H(\tau,2)$, each one exactly $\frac{|C|}{2^\tau}$ times.

The maximal τ with this property is called *strength* of the array.

BOA $C : (\tau, n, M)$, $M = |C|$.

Distance distributions of BOA

Definition 1. If $C \subset H(n,2)$ is a (τ, n, M) BOA and $x \in C$, then the $(n+1)$-tuple $(p_0(x), p_1(x), \ldots, p_n(x))$, where

$$p_i(x) = |\{y \in C : d(x,y) = i\}|$$

for $i = 0, 1, \ldots, n$, is called distance distribution of C with respect to x (or shorter: of x).

Fact 1. For many (relatively small) parameters all possible distance distributions can be calculated

Fact 2. There are connections between the distance distributions of related BOAs

Algorithms A and B for distance distributions

Algorithm A:

(1) Connection between $C = (\tau, n, M)$ and $C' = (\tau, n-1, M)$. We cut one column and use the relation between the distance distributions of points in C and C'

$$\underbrace{00\ldots0}_{n} \to \underbrace{00\ldots0}_{n-1}$$

We check fixed distribution in C against all possible distributions in C' – if no distribution of C' to fit then that distribution of C is ruled out.

(2) Apply (1) for the sequence (τ, τ, M), $(\tau, \tau+1, M)$, ..., $(\tau, n-1, M)$, (τ, n, M) starting from below and rule out some distance distributions

Algorithms A and B for distance distributions

Other results from algorithm A:

– the numbers x_i and y_i (the number of 0's and 1's, respectively, in the intersection of the cut column and the i-block; the i-block of C, $i \in \{0, 1, \dots, n\}$, is the set of all rows of weight i).

– all possibilities for the type of the cut column

Algorithms A and B for distance distributions

Algorithm B:

Similar to Algorithm A but with simultaneous cut of several columns which are determined by the support of some point of C

Uses connections between C and the derived C''

We check fixed distribution in C against all possible distributions in C'' – if no distribution of C'' to fit then that distribution of C is ruled out.

Connections between (τ, n, M) and $(\tau - 1, n - 1, M/2)$ BOAs

0 ⋮ 0	$(\tau - 1, n - 1, \frac{M}{2})$
1 ⋮ 1	$(\tau - 1, n - 1, \frac{M}{2})$

Connections between (τ, n, M) and $(\tau-1, n-1, M/2)$ BOAs

Theorem 1. *a) The vector* $(y_0, y_1, \ldots, y_{n-1})$ *coincides with some distance distribution of a point in a BOA of parameters* $(\tau-1, n-1, M/2)$.
b) The vector $(x_1, x_2, \ldots, x_n)$ *represents some distance distribution of a point in a BOA of parameters* $(\tau-1, n-1, M/2)$ *where the entries are rearranged under certain rule. In particular, when* C *contains a row of weight one and the cut column corresponds to the support of that row, then* $(x_1, x_2, \ldots, x_n)$ *coincides with some distance distribution of a point in a BOA of parameters* $(\tau-1, n-1, M/2)$.

New algorithm (addition to Algorithm A)

(1) Calculate all possible distance distributions of the targeted BOA (τ, n, M). Apply Algorithm A to reduce them.

(2) Calculate all possible distance distributions of BOA of parameters $(\tau - 1, n - 1, M/2)$. Apply Algorithm A to reduce them.

(3) Fix a distance distribution of the targeted BOA (τ, n, M) and consider all distance distributions of its derived $(\tau, n - 1, M)$ BOA. Rule out all distance distributions of the derived BOA whose vectors $(y_0, y_1, \ldots, y_{n-1})$ from the results of (1) do not appear as results of (2)

(4) Rule out the considered distance distribution of the targeted BOA if the results of (3) contradict to the results of Algorithm A.

Application for $(8, 12, 1536)$ BOA and consequences

We describe how the above algorithm works in the case of BOA of parameters $(8, 12, 1536)$. The existence of such BOAs is mentioned as undecided in Table 12.3 from the book *Orthogonal Arrays: Theory and Applications* of Hedayat, Stuffken and Sloane.

We calculate all feasible distance distributions of (8,12,1536) and apply Algorithm A. This results in 5 remaining distance distributions, namely

$$\begin{aligned}
W_1 &= (1, 0, 36, 80, 135, 432, 168, 432, 135, 80, 36, 0, 1),\\
W_2 &= (1, 0, 38, 63, 198, 300, 336, 306, 177, 92, 18, 7, 0),\\
W_3 &= (1, 0, 39, 54, 234, 216, 462, 180, 261, 56, 27, 6, 0),\\
W_4 &= (1, 1, 29, 99, 114, 426, 210, 390, 141, 101, 17, 7, 0),\\
W_5 &= (1, 1, 30, 90, 150, 342, 336, 264, 225, 65, 26, 6, 0),
\end{aligned}$$

Application for $(8, 12, 1536)$ BOA and consequences

For every W_i we know all possible distance distributions of BOA of parameters (8,11,1536) which can appear after cut of a column and, moreover, we know how many times appears each of them. Explicitly, we have again 5 possibilities:

$$
\begin{aligned}
V_1 &= (1, 5, 59, 69, 354, 210, 462, 186, 141, 41, 7, 1),\\
V_2 &= (1, 6, 50, 105, 270, 336, 336, 270, 105, 50, 6, 1),\\
V_3 &= (1, 7, 41, 141, 186, 462, 210, 354, 69, 59, 5, 1),\\
V_4 &= (1, 8, 33, 168, 138, 504, 210, 312, 117, 32, 13, 0),\\
V_5 &= (2, 0, 60, 120, 180, 504, 168, 360, 90, 40, 12, 0),
\end{aligned}
$$

Application for $(8, 12, 1536)$ BOA and consequences

All feasible distance distributions of $(7, 11, 768)$ BOA

$$
\begin{aligned}
U_1 &= (1, 0, 30, 60, 90, 252, 84, 180, 45, 20, 6, 0),\\
U_2 &= (1, 0, 31, 52, 118, 196, 154, 124, 73, 12, 7, 0),\\
U_3 &= (1, 0, 32, 45, 138, 168, 168, 138, 45, 32, 0, 1),\\
U_4 &= (1, 1, 25, 65, 110, 182, 182, 110, 65, 25, 1, 1),\\
U_5 &= (1, 2, 18, 85, 82, 196, 196, 82, 85, 18, 2, 1),
\end{aligned}
$$

3.1. For W_1 Algorithm A gives V_1, V_2 and V_3 as possibilities but V_3 has vector $(y_0, y_1, \ldots, y_{11})$ which is not in the list $U_1, \ldots, U_5$. Now V_1 and V_2 correspond to a unique solution which shows that V_1 can not appear but V_2 appears after cut of every column of the targeted (8,12,1536). Thus W_1 remains for further consideration.

Application for $(8, 12, 1536)$ BOA and consequences

3.2. For W_2 Algorithm A gives V_1, V_2 and V_4 as possibilities but V_1 has vector $(y_0, y_1, \ldots, y_{11})$ which is not in the list $U_1, \ldots, U_5$. On the other hand, V_1, V_2 and V_4 correspond to a unique solution which shows that all of them must appear after cut of a column of the targeted (8,12,1536). Therefore W_2 is ruled out.

3.3. For W_3 Algorithm A gives V_1, V_2, V_3 and V_4 as possibilities. Now V_1 and V_2 have vectors $(y_0, y_1, \ldots, y_{11})$ which are not in the list $U_1, \ldots, U_5$. On the other hand, V_1, V_2, V_3 and V_4 correspond to a unique solution which shows that V_1 must appear after cut of 6 columns of the targeted (8,12,1536). Therefore W_3 is ruled out.

Application for $(8, 12, 1536)$ BOA and consequences

3.4. For W_4 Algorithm A gives V_1, V_2, V_4 and V_5 as possibilities. Now V_2, V_4 and V_5 have vectors $(y_0, y_1, \dots, y_{11})$ which are not in the list $U_1, \dots, U_5$. On the other hand, V_1, V_2, V_4 and V_5 correspond to a unique solution which shows that all of them must appear after cut of a column of the targeted (8,12,1536). Therefore W_5 is ruled out.

3.5. For W_5 Algorithm A gives V_1, V_2, V_3, V_4 and V_5 as possibilities but V_1, V_3 and V_4 have vectors $(y_0, y_1, \dots, y_{11})$ which are not in the list $U_1, \dots, U_5$. On the other hand, V_1, V_2, V_3, V_4 and V_5 correspond to a unique solution which shows that V_1 and V_4 must appear after cut of a column of the targeted (8,12,1536). Therefore W_5 is ruled our.

Application for $(8, 12, 1536)$ BOA and consequences

Only W_1 remains as possible distance distribution in $(8, 12, 1536)$. Algorithm B with $\tau_0 = 3$ implies that the points of weight 3 in such a BOA must have distance distribution W_3, which was ruled out above.

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Thank you for your attention!