Seventh International Workshop on Optimal Codes and Related Topics
September 6-12, 2013, Albena, Bulgaria *pp. 181-186*

Local distributions of q-ary eigenfunctions and of q-ary perfect colorings

Anastasia Vasil'eva vasilan@math.nsc.ru
Sobolev Institute of Mathematics,
Novosibirsk State University, Novosibirsk, RUSSIA

Dedicated to the memory of Professor Stefan Dodunekov

Abstract. In the paper we investigate the eigenfunctions and perfect colorings on the graph of n-dimensional q-ary Hamming space. First, we obtain the interdependence of local distributions of an eigenfunction in two orthogonal faces. Then, we prove the analogous result for perfect colorings.

1 Introduction

We study eigenfunctions and perfect colorings of n-dimensional q-ary hypercube. The aim of the paper is to provide an explicit formula of connection for local distributions in two orthogonal faces. Earlier this question was considered in [2,4–6] for 1-error correcting perfect codes and perfect colorings in binary case ($q = 2$). In case $q > 2$ the question is investigated in [1] for 1-error-correcting codes. On the other hand in [3] a more general case of direct product of graphs is studied; however, the formula is not extended for classes of graphs.

Note that completely regular codes which are extensively investigated are particular case of perfect colorings.

This paper was inspired by [1]. The paper is organised as follows. In Section 2 we give necessary notations and propositions. In Section 3 we establish the formula for local weight enumerators of eigenfunctions in a pair of orthogonal faces. Using this formula we obtain in Section 4 the formula for local weight enumerator of perfect colorings in a pair of orthogonal faces. Remark that derived formulas are symmetric under choice of the face from the pair.

2 Preliminaries

Consider the set $\mathbf{F}_q = \{0, 1, \ldots, q-1\}$ as the group by modulo q and the hypercube $\mathbf{F}_q^n$ as the abelian group $\mathbf{F}_q \times \ldots \times \mathbf{F}_q$. We investigate functions and colorings on the graph $\mathbf{F}_q^n$ of n-dimensional q-ary hypercube, in this graph two vertices are adjacent iff the Hamming distance between them equals 1.

Take a vertex $\alpha \in \mathbf{F}_q^n$. Here and elsewhere I denotes a subset of $\{1, \ldots, n\}$ and $\overline{I} = \{1, \ldots, n\} \backslash I$. Denote by $s(\alpha)$ the support of a vertex α, i.e. the set of nonzero positions of α; the cardinality of the support is equal to the Hamming weight of α. Write $W_i(\alpha)$ for the set of all vertices that differ from α in i positions. By definition, put

$$\Gamma_I(\alpha) = \{\beta \in \mathbf{F}_q^n \ : \ \beta_i = \alpha_i \ \forall \ i \notin I\},$$

then $\Gamma_I(\alpha)$ is said to be a $(n - |I|)$-dimensional face, it has a structure of $\mathbf{F}_q^{n-|I|}$. Write simply W_i and Γ_I instead of $W_i(\alpha)$ and $\Gamma_I(\alpha)$ in case α is all-zero vertex. We say that two faces $\Gamma_I(\alpha)$ and $\Gamma_J(\beta)$ are orthogonal if $I \bigcap J = \emptyset$ and $I \bigcup J = \{1, \ldots, n\}$. It is easy to see that orthogonal faces have exactly one common vertex.

For any $\alpha, \beta \in \mathbf{F}_q^n$ define $\langle \alpha, \beta \rangle = \alpha_1\beta_1 + \ldots + \alpha_n\beta_n (\mod q)$. Consider the set of all functions $f : \mathbf{F}_q^n \longrightarrow \mathbb{C}$ as vector space V over $\mathbb{C}$. Let $\xi = e^{2\pi i/q}$. A function

$$\varphi^\beta(\alpha) = \xi^{\langle \alpha, \beta \rangle}, \qquad \alpha, \beta \in \mathbf{F}_q^n,$$

is called the character. The characters φ^β, $\beta \in \mathbf{F}_q^n$, forms the orthogonal basis of the vector space V. Define Fourier transform $\widehat{f}$ of a function f as follows:

$$\widehat{f(\alpha)} = \sum_{\beta \in \mathbf{F}_q^n} f(\beta)\varphi^\beta(\alpha) = \sum_{\beta \in \mathbf{F}_q^n} f(\beta)\xi^{\langle \alpha, \beta \rangle}, \quad \alpha \in \mathbf{F}_q^n. \tag{1}$$

Here we state necessary technical lemmas.

Lemma 1. *Let* $b \in \{0, 1, \ldots, q-1\}$. *Then*

$$\sum_{a=0}^{q-1} \xi^{ab} t^{|s(a)|} = \begin{cases} 0, & b \neq 0 \\ 1 + (q-1)t, & b = 0 \end{cases}$$

Lemma 2. *Let* $\beta \in \mathbf{F}_q^n$. *Then*

$$\sum_{\alpha \in \Gamma_I} \xi^{\langle \alpha, \beta \rangle} t^{|s(\alpha)|} = (1-t)^{|I \bigcap s(\beta)|}(1 + (q-1)t)^{|I| - |I \bigcap s(\beta)|}$$

Proof. By definition of character,

$$\sum_{\alpha \in \Gamma_I} \xi^{\langle \alpha, \beta \rangle} t^{|s(\alpha)|} = \sum_{\alpha_{i_1}=0}^{q-1} \cdots \sum_{\alpha_{i_k}=0}^{q-1} \prod_{j=1}^{k} \xi^{\alpha_{i_j}\beta_{i_j}} t^{|s(\alpha_{i_j})|}$$

Change the order of summations and multiplication, then apply Lemma 1:

$$= \prod_{j=1}^{k} \sum_{\alpha_{i_j}=0}^{q-1} \xi^{\alpha_{i_j}\beta_{i_j}} t^{|s(\alpha_{i_j})|} = (1-t)^{|I \bigcap s(\beta)|}(1 + (q-1)t)^{|I| - |I \bigcap s(\beta)|}$$

□

3 Eigenfunctions

The first object are eigenfunctions of the n-dimensional q-ary hypercube $\mathbf{F}_q^n$. It is known that the eigenvalues λ of the graph of n-dimensional q-ary hypercube are equal to $(q-1)n - qi,\ i = 0, 1, \ldots, n$. The corresponding eigenfunctions (we call them λ-functions) satisfy an equation

$$\sum_{\beta \in W_1(\alpha)} f(\beta) = ((q-1)n - qi) f(\alpha), \quad \alpha \in \mathbf{F}_q^n. \tag{2}$$

Rewrite this equations in a matrix form. Let D be the adjacency matrix of $\mathbf{F}_q^n$. Then

$$Df = \lambda f,$$

here f is a vector of values of the function. It is easy to see that Fourier coefficients of a λ-function f equal zero apart from the case where α has the Hamming weight $i = ((q-1)n - \lambda)/q$.

Now we introduce the concept of a local distribution. By definition, put

$$v_j^{I,f}(\alpha) = \sum_{\beta \in \Gamma_I(\alpha) \bigcap W_j(\alpha)} f(\beta),$$

the vector $v^{I,f}(\alpha) = (v_0^{I,f}(\alpha), \ldots, v_{|I|}^{I,f}(\alpha))$ is called the local distribution of the function f in the face $\Gamma_I(\alpha)$ with respect to the vertex α, or shortly (I,α)-local distribution of f. We say that

$$g_f^{I,\alpha}(x,y) = \sum_{j=0}^{k} v^{I,f}(\alpha)_j y^j x^{k-j} = \sum_{\beta \in \Gamma_I(\alpha)} f(\beta) y^{|s(\beta)|} x^{|I| - |s(\beta)|}$$

is a local weight enumerator. For simplicity of notation we omit α if it is the all-zero vertex.

Lemma 3. *Let f be an arbitrary function. Then*

$$g_f^I(x,y) = q^{-n} \sum_{\beta \in \mathbf{F}_q^n} \widehat{f}(\beta)(x + (q-1)y)^{|I| - |I \bigcap s(\beta)|}(x-y)^{|I \bigcap s(\beta)|}$$

Proof. follows from Lemma 2. □

Now we are ready to derive the relationship between local weight enumerators in two orthogonal faces.

Theorem 1. *Let λ be an eigenvalue of $\mathbf{F}_q^n$, f be a λ-function, $h = \frac{(q-1)n - \lambda}{q}$ and $\alpha \in \mathbf{F}_q^n$. Then*

$$(x + (q-1)y)^{h - |\overline{I}|} g_f^{\overline{I},\alpha}(x,y) = (x' + (q-1)y')^{h - |I|} g_f^{I,\alpha}(x', y'),$$

where $x' = x + (q-2)y,\ y' = -y$.

Proof. First note that the faces $\Gamma_I(\alpha)$ and $\Gamma_{\overline{I}}(\alpha)$ are orthogonal. Without loss of generality assume that α is all-zero vertex. Using lemma 3 write the local weight enumerator $g_f^{\overline{I}}(x,y)$ of an arbitrary function f in the orthogonal face $\Gamma_{\overline{I}}$:

$$q^{-n}\sum_{\beta\in\mathbf{F}_q^n}\widehat{f}(\beta)(x+(q-1)y)^{n-|I|-|s(\beta)|+|I\bigcap s(\beta)|}(x-y)^{|s(\beta)|-|I\bigcap s(\beta)|} \tag{3}$$

Since $\widehat{f}(\beta)=0$ for any $\beta\notin W_h$, then the summation in (3) is taken over all vertices of weight h (instead of all vertices of $\mathbf{F}_q^n$). This implies that

$$g_f^{\overline{I}}(x,y)=q^{-n}(x+(q-1)y)^{n-|I|-h}(x-y)^{h-|I|}\times$$
$$\times\sum_{\beta\in W_h}\widehat{f}(\beta)(x+(q-1)y)^{|I\bigcap s(\beta)|}(x-y)^{|I|-|I\bigcap s(\beta)|}.$$

Choose new variables x' and y' such that

$$x'+(q-1)y'=x-y,\qquad x'-y'=x+(q-1)y.$$

Hence,

$$g_f^{\overline{I}}(x,y)=q^{-n}(x+(q-1)y)^{n-|I|-h}(x-y)^{h-|I|}\times$$
$$\times\sum_{\beta\in W_h}\widehat{f}(\beta)(x'-y')^{|I\bigcap s(\beta)|}(x'+(q-1)y')^{|I|-|I\bigcap s(\beta)|}$$

Comparing with Lemma 3, finally get

$$g_f^{\overline{I}}(x,y)=(x+(q-1)y)^{n-|I|-h}(x-y)^{h-|I|}g_f^I(x',y')$$

□

4 Perfect colorings

In this section we prove an analog of Theorem 1 for perfect colorings. The partition $C=(C_1,\dots,C_r)$ of $\mathbf{F}_q^n$ is called a perfect coloring (or an equitable partition) with the parameter matrix $S=(s_{ij})_{i,j=1,\dots,r}$ if for any $i,j\in\{1,\dots,r\}$ and any vertex $\alpha\in C_i$ the number of vertices $\beta\in C_j$ at distance 1 from α is equal to s_{ij}. First present a perfect r-coloring by $(0,1)$-matrix C of size $q^n\times r$ that defined as follows: each row has only one nonzero position that marks the color of the corresponding vertex. Then the coloring is perfect iff

$$DC=CS, \tag{4}$$

where D is the adjacency matrix of $\mathbf{F}_q^n$.

Define a local distribution of a coloring as local distribution of characteristic functions of the colors. More precisely, a local distribution of the coloring C in the face $\Gamma_I(\alpha)$ with respect to the vertex α is $(r \times |I|)$-matrix

$$v^{I,C}(\alpha) = \begin{pmatrix} v_0^{I,C_1}(\alpha) & \dots & v_{|I|}^{I,C_1}(\alpha) \\ \vdots & & \vdots \\ v_0^{I,C_r}(\alpha) & \dots & v_{|I|}^{I,C_r}(\alpha) \end{pmatrix},$$

where $v_j^{I,C_i}(\alpha) = |C_i \bigcap W_j(\alpha) \bigcap \Gamma_I(\alpha)|, \quad i = 1, \dots, r, \ j = 0, \dots, |I|$. A vector-function

$$g_C^{I,\alpha}(x, y) = (g_{C_1}^{I,\alpha}(x, y), \dots, g_{C_r}^{I,\alpha}(x, y))$$

is called the local weight enumerator of the coloring C in the face $\Gamma_I(\alpha)$ with respect to the vertex α.

Now we claim an analog of Theorem 1 for perfect colorings.

Theorem 2. *Let $C = (C_1, \dots, C_r)$ be a perfect coloring of $\mathbf{F}_q^n$ with parameter matrix S and $\alpha \in \mathbf{F}_q^n$. Put $h(S) = \frac{(q-1)nE - S}{q}$. Then*

$$g_C^{\overline{I},\alpha}(x, y)(x + (q-1)y)^{h(S) - |\overline{I}|E} = g_C^{I,\alpha}(x', y')(x' + (q-1)y')^{h(S) - |I|E}. \quad (5)$$

Proof. Without loss of generality put $\alpha = (0, \dots, 0)$.

Perfect colorings are closely related with eigenfunctions. Indeed, let $\mu_1, \dots, \mu_r$ be eigenvalues and $T^1, \dots, T^r$ be eigenvectors of the parameter matrix S, i.e.

$$ST^i = \mu_i T^i, \quad i = 1, \dots, r.$$

Thus, it holds

$$ST = TM$$

for the matrices $T := [T^1, \dots, T^r]$ and $M = diag\{\mu_1, \dots, \mu_r\}$. Multiplying both sides of (4) by T and applying the last equation, we get

$$DCT = CST = CTM.$$

It means that columns of $F = CT$ are eigenfunctions of D, denote them by $F^1, \dots, F^r$. Then

$$DF = FM,$$

where $F = (F^1, \dots, F^r)$. By Theorem 2,

$$(x + (q-1)y)^{h_i - |\overline{I}|} g_{F^i}^{\overline{I}}(x, y) = (x' + (q-1)y')^{h_i - |I|} g_{F^i}^{I}(x', y'), \quad i = 1, \dots, r. \quad (6)$$

Put $g_F = (g_{F^1}, \dots, g_{F^r})$ and

$$M_I(x,y) = diag\left\{(x+(q-1)y)^{\frac{(q-1)n-\mu_1}{q}-|I|}, \dots, (x+(q-1)y)^{\frac{(q-1)n-\mu_r}{q}-|I|}\right\}.$$

Rewrite the equation (6) in terms of these matrix:

$$g_F^{\overline{I}}(x,y)M_{\overline{I}}(x,y) = g_F^I(x',y')M_I(x',y').$$

Note that

$$g_{F^i} = \sum_j T_j^i g_{C_j}, \quad g_F = g_C T.$$

Therefore we obtain

$$g_C^{\overline{I}}(x,y)TM_{\overline{I}}(x,y) = g_C^I(x',y')TM_I(x',y'). \tag{7}$$

By definition of a matrix function

$$TM_I(x,y)T^{-1} = (x+(q-1)y)^{\frac{(q-1)nE-S}{q}-|I|E}.$$

To conclude the proof it remains multiply both sides of (7) by T^{-1}. □

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