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Two types of upper bounds on the smallest size of a complete arc in the plane $PG(2, q)$

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Dedicated to the memory of Professor Stefan Dodunekov

Abstract. In the projective planes $PG(2, q)$, with the help of a computer search using randomized greedy algorithms, more than 5030 new small complete arcs are obtained for $q \in H = H_1 \cup H_2 \cup S$ where $H_1 = \{q : 19 \leq q \leq 44519,$ q is a power prime$\}$, $H_2 = \{q : 44531 \leq q \leq 66749,\ q \text{ prime}\}$, S is a set of 110 sporadic primes q with $67003 \leq q \leq 300007$. Using the new arcs, it is shown that for the smallest size $t_2(2, q)$ of a complete arc in $PG(2, q)$, $q \in H$, it holds that $t_2(2, q) < D\sqrt{q}(\ln q)^{\varphi(q;D)}$ where $\varphi(q; D)$ is a *decreasing function* of q, D is a constant independent of q, and $\varphi(q; 0.6) = 1.51/\ln q + 0.8028$. Also, by probabilistic methods it is shown that $t_2(2, q) < B(q) < 2\sqrt{q}\ln^{0.5} q$ where $B(q) = \lceil x \rceil$ while x is the solution of equation (7). The probabilistic bounds are confirmed by computer search for $q \in H$. Moreover, our results allow us to conjecture that all above mentioned upper estimates hold for all $q \geq 19$.

1 Introduction

Let $PG(2, q)$ be the projective plane over the Galois field F_q. An n-arc is a set of n points no three of which are collinear. An n-arc is called complete if it is not contained in an $(n + 1)$-arc of $PG(2, q)$.

In [4] the relationship among the theory of n-arcs, coding theory and mathematical statistics is presented. In particular, a complete arc in a plane $PG(2,q)$, the points of which are treated as 3-dimensional q-ary columns, defines a parity check matrix of a q-ary linear code with codimension 3, Hamming distance 4, and covering radius 2. Arcs can be interpreted as linear maximum distance separable (MDS) codes [4] and they are related to optimal coverings arrays and to superregular matrices.

One of the most important problems in the study of projective planes, which is also of interest in coding theory, is the determination of the smallest size $t_2(2,q)$ of a complete arc in $PG(2,q)$. This is a hard open problem. Surveys and results on the sizes of plane complete arcs can be found, e.g. in [1–6] and the references therein. The exact values of $t_2(2,q)$ are known only for $q \leq 32$ [6].

This work is devoted to *upper bounds* on $t_2(2,q)$.

Let $t(\mathcal{P}_q)$ be the size of the smallest complete arc in any (not necessarily Galois) projective plane $\mathcal{P}_q$ of order q. In [5], for *sufficiently large* q, the following result is proved by *probabilistic methods* (we give it in the form of [4]):

$$t(\mathcal{P}_q) \leq D\sqrt{q}\log^C q,\ C \leq 300,$$

where C and D are constants independent of q (i.e. so-called universal or absolute constants). The logarithm basis is not noted as the estimate is asymptotic. The authors of [5] conjecture that the constant can be reduced to $C = 10$. The smallest size of a complete arc in $PG(2,q)$ obtained via algebraic constructions is $cq^{3/4}$, where c is an universal constant.

By computer search, in [2] it is shown that in $PG(2,q)$, the following holds:

$$t_2(2,q) < 0.7\sqrt{q}(\ln q)^{\ln^{-1} q+0.7805}, \quad 23 \leq q \leq 13627;$$
$$t_2(2,q) < \sqrt{q}\ln^{0.72983} q, \quad 109 \leq q \leq 13627;$$
$$t_2(2,q) < 5.15\sqrt{q}, \quad q \leq 13627.$$

2 Upper bounds on $\bar{t}_2(2,q)$ based on a decreasing degree of logarithm of q

We denote the following sets of q values.

$H = H_1 \cup H_2 \cup S$, $H_1 = \{q : 19 \leq q \leq 44519,\ q \text{ is a power prime}\}$,
$H_2 = \{q : 44531 \leq q \leq 66749,\ q \text{ prime}\}$,
S is a set of 110 sporadic prime q with $67003 \leq q \leq 300007$, see Table 2.

The following **form of the upper bound** [2] is used:

$t_2(2,q) < D\sqrt{q}(\ln q)^{\varphi(q;D)}$ where $\varphi(q;D)$ is a *decreasing function* of q;
D is a constant independent of q.

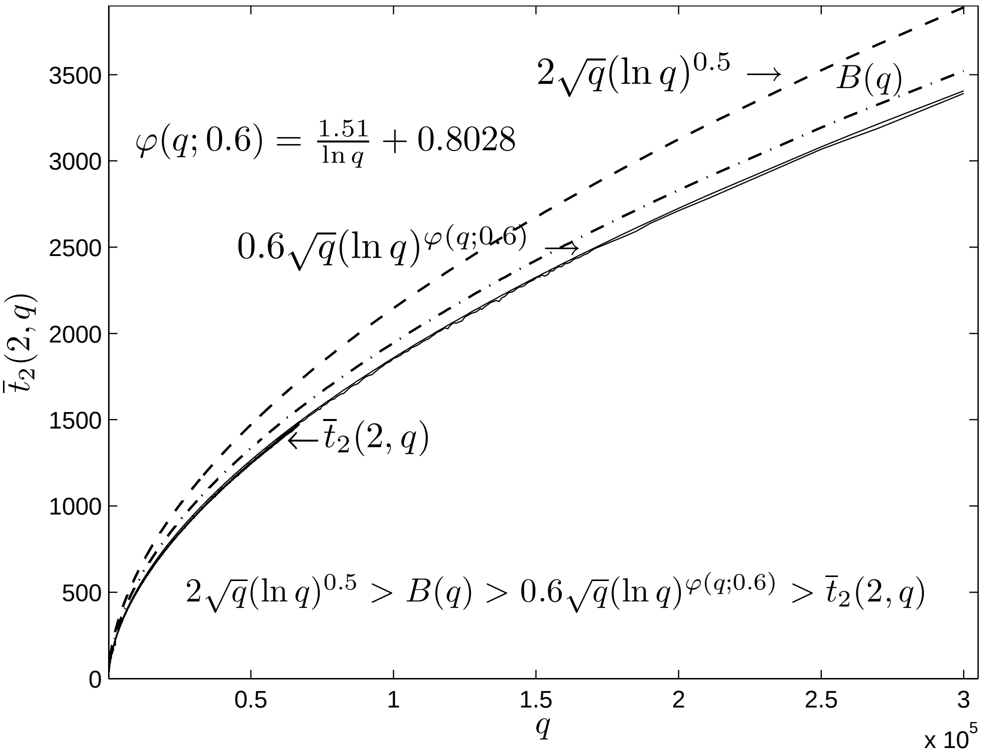

Figure 1: The values of $2\sqrt{q}(\ln q)^{0.5}$ (the top dashed curve), $B(q)$ (the 2-nd dashed-dotted curve), $0.6\sqrt{q}(\ln q)^{\varphi(q;0.6)}$ (the 3-rd curve), and $\bar{t}_2(2,q)$ (the bottom curve) for $q \in H$. $B(q) = \lceil x \rceil$ where x is the solution of equation (7). $\bar{t}_2(2,q)$ is the smallest *known* size of a complete arc in $PG(2,q)$. The curves $0.6\sqrt{q}(\ln q)^{\varphi(q;0.6)}$ and $\bar{t}_2(2,q)$ almost coalesce with each other.

For computer search we use the randomized greedy algorithms, see [1, 2]. In this work we show the following.

Theorem 1. *In $PG(2,q)$, the following holds:*

$$t_2(2,q) < 0.6\sqrt{q}(\ln q)^{\varphi(q;0.6)},\ \varphi(q;0.6) = \frac{1.51}{\ln q} + 0.8028,\ q \in H; \tag{1}$$

$$t_2(2,q) < \sqrt{q}\ln^{0.72959} q,\ 109 \le q \in H; \tag{2}$$

$$t_2(2,q) < 5.2\sqrt{q} \textit{ for } q \le 16369,\ q \ne 16249;$$

$$t_2(2,q) < 5.5\sqrt{q} \textit{ for } q \le 37321,\ q \ne 36481, 37249;$$

$$t_2(2,q) < 5.7\sqrt{q} \textit{ for } q \le 63803,\ q \in H,\ q \ne 61519, 62459, 62987.$$

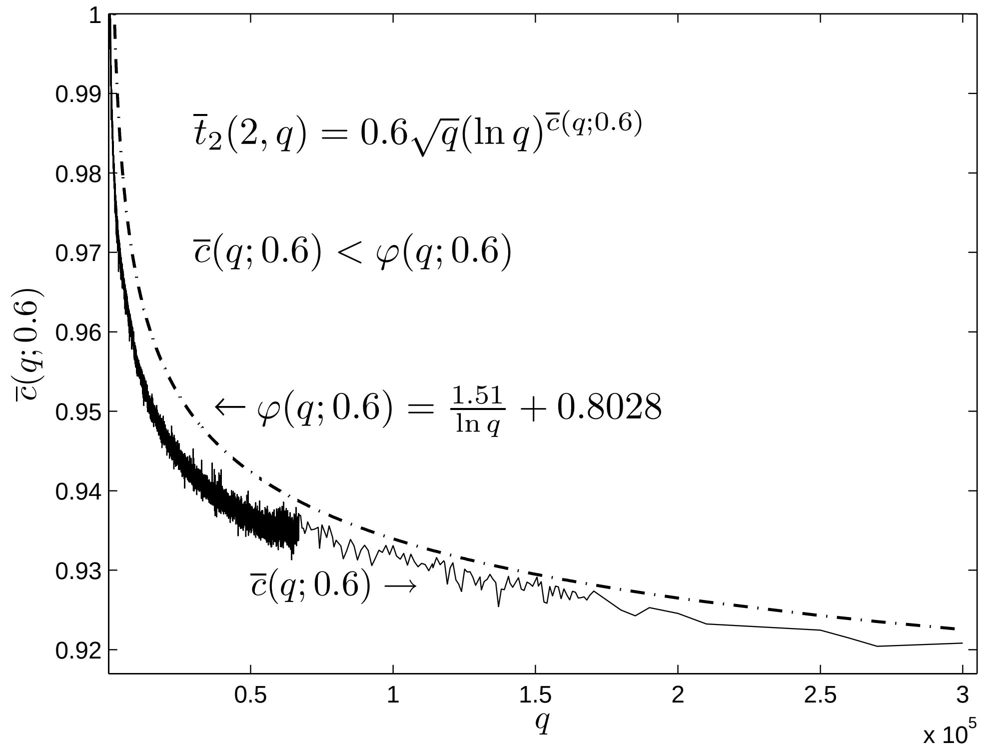

Figure 2: The values of $\varphi(q;0.6) = \frac{1.51}{\ln q} + 0.8028$ (the top dashed-dotted curve) and $\overline{c}(q;0.6)$ (the bottom curve) for $q \in H$. $\overline{c}(q;0.6)$ is given by the equality $\overline{t}_2(2,q) = 0.6\sqrt{q}(\ln q)^{\overline{c}(q;0.6)}$.

Conjecture 2. *The estimate* (1) *holds for all* $q \geq 19$ *and the estimate* (2) *holds for all* $q \geq 109$.

Let $\bar{t}_2(2,q)$ be the smallest *known* size of a complete arc in $PG(2,q)$.

In order to obtain the estimate (2) we improve (in comparison with [2]) the values of $\bar{t}_2(2,q)$ given in Table 1.

Table 1: New sizes $\bar{t}_2 = \bar{t}_2(2,q)$ of complete arcs in $PG(2,q)$ smaller than in [2]

q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$
2551	226	2861	241	3125	255	3167	257	3313	263	3463	271
3529	274	3571	274	3593	277	3637	279	3659	280	3727	283
3769	285	3793	286	3907	291	3929	292	4091	299	4231	305
4327	309	4349	310	4373	311	4397	312	4421	313	4493	315
4591	320	4639	322	4787	328	4937	333	5503	356	5531	356

On Figures 1 – 3 in regarding to their captions, a number of discussed values and functions are shown.

The values of $\bar{t}_2(2,q)$ for $q \in S$ are given in Table 2.

Table 2: The smallest known sizes $\bar{t}_2 = \bar{t}_2(2,q)$ of complete arcs in planes $PG(2,q)$, $q \in S$

q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$	q	$\bar{t}_2$
67003	1483	67511	1489	68023	1490	68501	1496	69001	1500	69539	1509
70001	1518	71249	1530	72503	1546	73721	1562	74201	1558	75011	1579
76733	1594	77509	1609	78713	1618	79201	1619	80021	1632	81701	1649
82507	1652	83701	1670	85009	1691	86257	1702	87509	1709	88721	1728
90001	1745	91229	1756	92503	1762	93701	1779	95003	1798	96211	1810
97501	1828	98711	1836	100003	1855	101203	1862	102503	1874	103703	1890
105019	1897	106207	1916	107507	1925	108761	1935	110017	1950	111253	1962
112501	1971	113759	1988	114001	1988	115259	2007	116507	2015	117751	2030
118033	2032	119027	2028	120011	2048	121001	2061	122011	2060	123001	2067
124001	2072	125101	2097	126011	2104	127031	2114	128021	2120	129001	2124
130513	2144	131009	2140	132001	2155	133013	2166	134033	2179	135007	2185
136013	2192	137029	2185	138007	2206	139021	2215	140521	2238	141023	2238
142007	2244	143053	2257	144013	2269	145007	2270	146009	2278	147011	2293
148013	2301	149011	2308	150503	2328	151007	2326	152003	2338	153001	2332
154001	2340	155003	2361	156007	2363	157007	2379	158003	2387	159013	2387
160001	2403	161009	2407	162007	2412	163003	2427	164011	2435	165001	2439
166013	2448	167009	2457	168013	2463	169003	2475	170503	2491	180001	2555
185021	2591	190027	2638	200003	2712	210011	2780	250007	3067	260003	3129
270001	3189	300007	3391								

3 Probabilistic upper bounds on $\bar{t}_2(2,q)$

We consider probabilistic algorithms, creating a complete arc step by step, e.g. the greedy algorithm [1, 2] or FOP algorithm with the lexicographical order of points [3]. After the i-th step, the corresponding arc contains i points.

Let $\theta = q^2+q+1$ be the number of points in $PG(2,q)$. Let $NONcov_i$ be the number of noncovered points of $PG(2,q)$ after the i-th step of the algorithm. Let $NEWcov_{i+1}$ be the number of points covered in first on the $i+1$-th step. On the $i+1$-th step, drawing lines through i old points (that have been included into an arc) and a one new point (that will be added to the arc) we obtain

$$CAND_{i+1} = i(q-1)$$

where $CAND_{i+1}$ is the number of candidates to be a new covered point.

For probabilistic algorithms, it is natural to conjecture that *candidates on covered and noncovered arias of the plane are distributed uniformly.* In other words: the **proportion** $\frac{NONcov_i}{\theta}$ of noncovered points is the **probability** that a random point of the plane is noncovered. This implies

$$NEWcov_{i+1} \cong 1 + CAND_{i+1} \times \frac{NONcov_i}{\theta}. \tag{3}$$

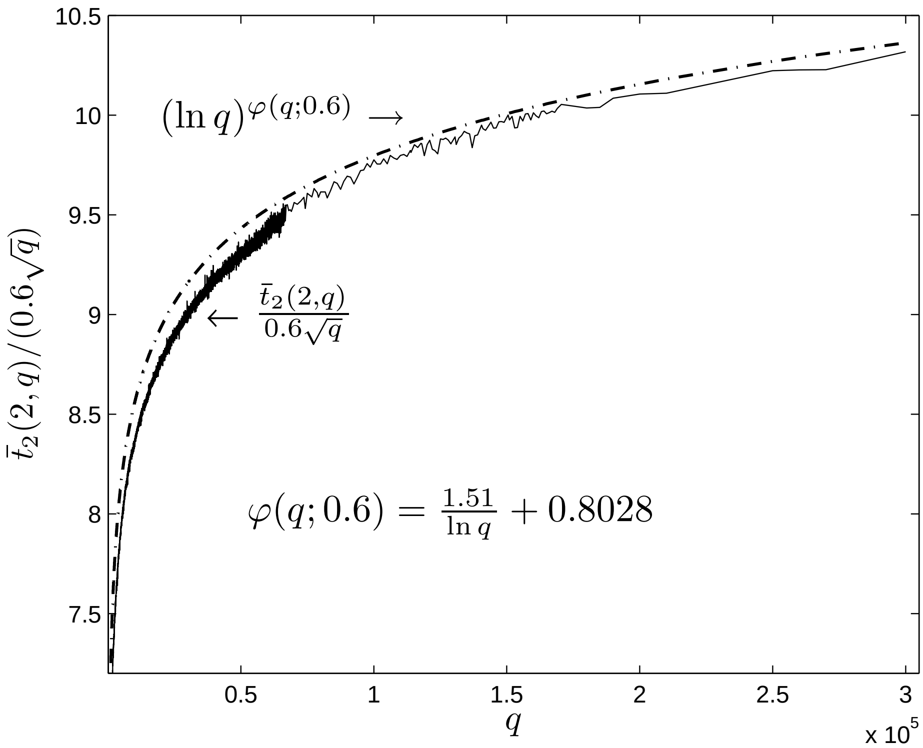

Figure 3: The values of $(\ln q)^{\varphi(q;0.6)}$ (the top dashed-dotted curve) and $\bar{t}_2(2,q)/(0.6\sqrt{q})$ (the bottom curve) for $q \in H$

So, we conjecture that there exists a noncovered point for which (3) holds. Moreover, on every step, a greedy algorithm can add to the arc a new point with the *maximal value* of $NEWcov_{i+1}$ in comparison with other possible new points. In principle, this can provide change of the sign "$\cong$" in (3) by "$\gtrsim$".

Lemma 3. *Under condition* (3)*, it holds that*

$$\frac{NONcov_{k+1}}{\theta} \cong f_q(k) - \frac{1}{\theta}\Delta_q(k); \tag{4}$$

$$f_q(k) = \prod_{i=1}^{k}(1 - \frac{i}{q}); \quad 1 < \Delta_q(k) = 1 + \sum_{j=1}^{k}\prod_{i=j}^{k}(1 - \frac{i}{q}) < k+1. \tag{5}$$

Theorem 4. *Under condition* (3)*, there are the following probabilistic bounds.*

$$t_2(2,q) < B(q) < 2\sqrt{q}\ln^{0.5} q \tag{6}$$

where $B(q) = \lceil x \rceil$ while x is the solution of equation (7) *of the form*

$$f_q(x) - \frac{1}{\theta}\Delta_q(x) = 0. \quad (7)$$

Proof. The bound $t_2(2,q) < B(q)$ follows from (4). Then, as $e^{-1/i} > 1 - 1/i$, we have $f_q(x) < e^{-x^2/2q}$ whence $x < \sqrt{2q}\sqrt{\ln \frac{q^2+q+1}{\Delta_q(x)}}$. Finally, by (5), we may put that $\frac{q^2+q+1}{\Delta_q(x)} < q^2$ whence $x < 2\sqrt{q}\ln^{0.5} q$. □

The "pessimistic" estimate $t_2(2,q) < 2\sqrt{q}\ln^{0.5} q$ is worse than $t_2(2,q) < B(q)$ but certainly $2\sqrt{q}\ln^{0.5} q$ is an upper bound, see Figure 1.

The probabilistic bounds (6) are confirmed by computer search for $q \in H$, see Figure 1 where solutions $B(q)$ of equation (7) are obtained by direct substitution of possible values of x. Note that for $q \in H$, we have $B(q) < 1.8186\sqrt{q}\ln^{0.5} q$ and $B(q) < 6.95\sqrt{q}\ln^{-2.52/\ln q+0.17} q$, i.e. the curves $1.8186\sqrt{q}\ln^{0.5} q$ and $6.95\sqrt{q}\ln^{-2.52/\ln q+0.17} q$ also are upper bounds for $t_2(2,q)$.

Conjecture 5. *The probabilistic bounds* (6) *hold for all* $q \geq 7$.

A part of the results of the work was obtained using computational resources of Multipurpose Computing Complex of National Research Centre "Kurchatov Institute" (http://computing.kiae.ru/).

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