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Blocking sets in finite projective spaces and the extension problem for linear codes

Ivan Landjev i.landjev@nbu.bg
New Bulgarian University, 21 Montevideo str., 1618 Sofia, BULGARIA
Assia Rousseva
Sofia University, Faculty of Mathematics and Informatics
5 James Bourchier Blvd., 1126 Sofia, BULGARIA

Dedicated to the memory of Professor Stefan Dodunekov

Abstract. We prove a new sufficient condition for the extendability of Griesmer arcs depending on the possible spectra of a maximal hyperplane.

1 Preliminaries

The geometric nature of certain problems in coding theory has been long known. In this paper we present a new result on the extendability of arcs in finite projective spaces which translates in a natural way into a result about the extendability of linear codes.

It is a well-known fact that adding a parity check to a binary $[n, k, d]$-code of odd minimum distance d increases the minimum distance of the codes, i.e. the resulting codes has parameters $[n+1, k, d+1]$. This result has been generalized by Hill and Lizak in [2, 3]. They showed that if all weights in an $[n, k, d]_q$ code are congruent to 0 or $d \pmod q$, with $(d, q) = 1$, then it can be extended to an $[n+1, k, d+1]_q$ code. It turned out that this fact has a natural explanation in terms of blocking sets containing a hyperplane. It was proved in [4, 6] that this result can be obtained from the well-known Bose-Burton theorem for blocking sets in $\mathrm{PG}(k-1, q)$. This result was further generalized in [5] by using a result of Beutelspacher and Heim on the size of the minimal non-trivial (i.e. not containing a hyperplane) blocking set in a finite projective geometry.

In a series of papers, Maruta obtained further results [6, 7, 8] on extendability of linear codes. He introduced the notion of diversity of a linear code with spectrum (A_i) as the pair (Φ_0, Φ_1), where

$$\Phi_0 = \frac{1}{q-1} \sum_{q|i, i\neq 0} A_i, \Phi_1 = \frac{1}{q-1} \sum_{i \not\equiv 0, d \pmod q} A_i,$$

He proved that for various values of the diversity this code is indeed extendable.

It has been proved in [1] that linear $[n,k,d]_q$-codes of full length and $(n,n-d)$-arcs in $\mathrm{PG}(k-1,q)$ are in some sense equivalent objects. With each linear code one can associate an arc (possibly in an non-unique way) so that semilinearly isomorphic codes give rise to equivalent arcs and vice versa. Arcs associated with codes meeting the Griesmer bound are called Griesmer arcs.

2 Basic definitions

Let $\mathcal{P}$ be the set of points of the projective geometry $\mathrm{PG}(k-1,q)$. Every mapping $\mathcal{K}:\mathcal{P}\to\mathbb{N}_0$ from the set of points of the geometry to the non-negative integers is called a multiset in $\mathrm{PG}(k-1,q)$. This mapping is extended additively to the subsets of $\mathcal{P}$: for every $\mathcal{Q}\subseteq\mathcal{P}$, $\mathcal{K}(\mathcal{Q})=\sum_{P\in\mathcal{Q}}\mathcal{K}(P)$. The integer $n:=\mathcal{K}(\mathcal{P})$ is called the cardinality of $\mathcal{K}$.

Multisets can be viewed as arcs or as blocking sets. A multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called an (n,w)-multiarc (or simply (n,w)-arc) if (1) $\mathcal{K}(\mathcal{P})=n$, (2) $\mathcal{K}(H)\le w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0)=w$. Similarly, a multiset $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called an (n,w)-blocking set (or (n,w)-minihyper) if (1) $\mathcal{K}(\mathcal{P})=n$, (2) $\mathcal{K}(H)\ge w$ for every hyperplane H, and (3) there exists a hyperplane H_0 with $\mathcal{K}(H_0)=w$.

An (n,w)-arc $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called extendable (or incomplete), if there exists an $(n+1,w)$-arc $\mathcal{K}'$ in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}'(P)\ge\mathcal{K}(P)$ for every point $P\in\mathcal{P}$. An arc is said to be complete if it is not extendable. Similarly, an (n,w)-blocking set $\mathcal{K}$ in $\mathrm{PG}(k-1,q)$ is called reducible, if there exists an $(n-1,w)$-blocking set $\mathcal{K}'$ in $\mathrm{PG}(k-1,q)$ with $\mathcal{K}'(P)\le\mathcal{K}(P)$ for every point $P\in\mathcal{P}$. A blocking set is called irreducible if it is not reducible.

3 Extendability of Griesmer arcs

Let $\mathcal{K}$ be an (n,w)-arc in $\Sigma=\mathrm{PG}(k-1,q)$. Assume that all multiplicities of hyperplanes in Σ are congruent to $n,n+1,\ldots,n+t$ modulo q for some constant $t<q$. This is a typical situation when one investigates the existence of Griesmer arcs with given parameters.

Define a new multiset $\overline{\mathcal{K}}$ in the dual geometry $\overline{\Sigma}$ by

$$\overline{\mathcal{K}}:\begin{cases}\mathcal{H} & \to\quad\mathbb{N}_0\\ H\to\overline{\mathcal{K}}(H)=n+t-\mathcal{K}(H)\pmod q. & \end{cases}\tag{1}$$

In other words, hyperplanes of multiplicity $n+a \pmod q$ become $(t-a)$-points in the dual geometry. The following result is straightforward.

Theorem 1. Let $\mathcal{K}$ an (n,w)-arc in $\Sigma=\mathrm{PG}(k-1,q)$ and let $\overline{\mathcal{K}}$ be defined by (1). If $\overline{\Sigma}$ contains a hyperplane without 0-points then $\mathcal{K}$ is extendable.

Proof. Since maximal hyperplanes correspond to 0-points in the dual geometry, the condition of the theorem is that there exists a point in Σ which is not incident with maximal hyperplanes. □

By Theorem 1, the extendability of arcs is linked with the structure of a certain multiset defined in the dual geometry. It turns out that this multiset is highly divisible.

Theorem 2. Let S be subspace of $\overline{\Sigma}$. Then

$$\overline{\mathcal{K}}(S) \equiv t \pmod q.$$

Proof. Let S be a line. It corresponds to a subspace Δ of codimension 2 in Σ. Denote by H_i, $i = 0, \ldots, q$ the set of all hyperpalnes through Δ. We have

$$n = \sum_{i=0}^{q} \mathcal{K}(H_i) - q\mathcal{K}(\Delta).$$

Reducing both sides modulo q and using the fact that $\mathcal{K}(H_i) + \overline{\mathcal{K}}(H_i) = n + t$, one gets

$$(q+1)(n+t) - \sum_{i=0}^{q} \overline{\mathcal{K}}(H_i) \equiv n \pmod q,$$

whence

$$\overline{\mathcal{K}}(S) = \sum_{i=0}^{q} \overline{\mathcal{K}}(H_i) \equiv t \pmod q.$$

For subspaces of larger dimension, we can use the fact that the multiplicity of each line L in S is $t \pmod q$ and sum the multiplicities of all lines through a fixed point in S. □

By the above theorem, the multiset $\overline{\mathcal{K}}$ has the following properties:

- the multiplicity of each point is at most t;

- the multiplicity of each subspace of dimension r, $1 \leq r \leq k-1$, is at least tv_r.

Here we use the conventional notation $v_r = (q^r - 1)/(q-1)$. In the general case, we do not know the cardinality of $\overline{\mathcal{K}}$.

For $t = 1$, the arc $\mathcal{K}$ is always extendable. In fact, this is another formulation of the theorem by Hill and Lizak. In this case the arc $\overline{\mathcal{K}}$ is projective. Further every line is 1- or $(q+1)$-line. Hence in the plane case $\overline{\mathcal{K}}$ is either a line, or the complete plane. In higher dimensions, one can easily check that $\overline{\mathcal{K}}$ is either a hyperplane or the complete space. In both cases there exists a hyperplane without 0-points, which implies that $\mathcal{K}$ is extendable by Theorem 1.

For $t = 2$ and odd $q \geq 5$, the arcs $\overline{\mathcal{K}}$ have been characterized by Maruta [6]. It turns out that in this case, the arc $\overline{\mathcal{K}}$ contains a hyperplane without 0-points, and the arc $\mathcal{K}$ is again extendable.

Theorem 3. Let $\mathcal{K}$ be a Griesmer $(n, w = n - d)$-arc with $\mathcal{K}(H) \equiv n, n + 1, \dots, n + t \pmod{q}$ for every hyperplane H. Denote by (a_i) the spectrum of the arc $\mathcal{K}|_{H_0}$, where H_0 is a hyperplane of multiplicity w, with respect to $\mathcal{K}$. Let A be the largest integer such that a $(tv_{k-1} + A, tv_{k-2})$-minihyper contains a hyperplane in its support. If

$$qa_{w-\lceil d/q\rceil-1} + 2qa_{w-\lceil d/q\rceil-2} + \dots + (t-1)q \sum_{u \leq w-\lceil d/q\rceil-t+1} a_u \leq A,$$

then $\mathcal{K}$ is extendable.

Proof. By the fact that $\mathcal{K}$ is a Griesmer arc, we have that

$$n = \sum_{i=0}^{k-1} \lceil \frac{d}{q^i} \rceil, \ w = \sum_{i=1}^{k-1} \lceil \frac{d}{q^i} \rceil.$$

The maximal multiplicity of a subspace of codimension 2 contained in H_0 is then

$$w' = w - \lceil \frac{d}{q} \rceil = \sum_{i=2}^{k-1} \lceil \frac{d}{q^i} \rceil.$$

Let $\overline{\mathcal{K}}$ be the arc in $\overline{\Sigma}$ defined earlier in this section. The point $x^* = H_0$ is a 0-point in $\overline{\Sigma}$. Denote by L_i^* all lines in $\overline{\Sigma}$ through x^*. They correspond to the hyperlines δ_i in H_0, i.e. the subspaces of codimension 2 that are contained in H_0.

Consider a fixed line $L^* = \delta$ where $\mathcal{K}(\delta) = w' - \lambda$, $\lambda \in \{0, \dots, t-2\}$. Denote by $H_0, H_1, \dots, H_q$ all hyperplanes through δ. Set

$$\mathcal{K}(H_i) = w - \alpha_i q - \beta_i, \ \ \beta_i \in \{0, \dots, t\},$$

Since $\mathcal{K}(H_i) + \overline{\mathcal{K}}(H_i) \equiv n + t \equiv w \pmod{q}$, we get that $\overline{\mathcal{K}}(H_i) = \beta_i$. Now we have

$$\begin{aligned} n &= \sum_{i=0}^{q} \mathcal{K}(H_i) - q(w' - \lambda) \\ &= \sum_{i=0}^{q} (w - \alpha_i q - \beta_i) - q(w' - \lambda) \\ &= w - q\sum_{i=0}^{q} \alpha_i - \sum_{i=0}^{q} \beta_i + q\lceil \frac{d}{q} \rceil + q\lambda, \end{aligned}$$

whence

$$\sum_{i=0}^{q} \beta_i = q\lceil \frac{d}{q} \rceil + q\lambda - d - q \sum_{i=0}^{q} \alpha_i.$$

Since $d \equiv -t \pmod q$, we have $q\lceil \frac{d}{q} \rceil - d = t$. This gives an upper bound on the multiplicity of L^* with respect to $\overline{\mathcal{K}}$

$$\overline{\mathcal{K}}(L^*) = \sum_{i=0}^{q} \beta_i = t + q\lambda - q \sum_{i=0}^{q} \alpha_i \leq t + q\lambda.$$

Now summing up the multiplicities of all lines L^* through x^* and taking into account that $\overline{\mathcal{K}}(x^*) = 0$ one gets

$$\begin{aligned} |\overline{\mathcal{K}}| &= \sum_i \overline{\mathcal{K}}(L_i^*) \\ &\leq a_{w'}t + a_{w'-1}(t+q) + \cdots + a_{w'-(t-2)}(t+(t-2)q) + \sum_{u \leq w'-(t-1)} a_u(t+(t-1)q) \\ &= \left(\sum_{u \leq w'} a_u \right) t + a_{w'-1}q + \cdots + a_{w'-(t-2)}(t-2)q + \sum_{u \leq w'-(t-1)} a_u(t-1)q \\ &= v_{k-1}t + a_{w'-1}q + \cdots + a_{w'-(t-2)}(t-2)q + \sum_{u \leq w'-(t-1)} a_u(t-1)q. \end{aligned}$$

If

$$a_{w'-1}q + \cdots + a_{w'-(t-2)}(t-2)q + \sum_{u \leq w'-(t-1)} a_u(t-1)q \leq A$$

we have that $|\overline{\mathcal{K}}| \leq tv_{k-1} + A$. This implies that $\overline{\mathcal{K}}$ contains a hyperplane without 0-points. Hence $\mathcal{K}$ is extendable by Theorem 1. □

The idea of Theorem 3 can be used to restrict the spectrum not only of the maximal hyperplanes, but also of hyperplanes with a smaller multiplicity.

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