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On integer code correcting single error of type $(\pm 1, 2)$

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Dedicated to the memory of Professor Stefan Dodunekov

1 Introduction

Nonvolatile memory is computer memory that maintains stored information without a power supply. Flash memory is currently the dominant nonvolatile memory because it is cheap and can be electrically programmed and erased with relative ease. Flash memory devices can be found almost everywhere nowadays. They are lighter, faster and more shock resistant than traditional magnetic hard drives.

The storage density of flash memory device is dependent on the number of the discrete voltage levels and thus store a single bit. The demand for increased storage capacity has created the need to store more than a single bit per cell by simply representing more than two voltage levels. As this technology scales and the storage density increases, data errors become more prevalent, making error correction coding critical for maintaining data integrity.

Flash devices exhibit a multitude of complex error types and behaviors, but common to all flavors of flash storage is the inherent asymmetry between cell programming (charge replacement) and cell erasing (charge removal). This asymmetry causes significant error sources to change cell levels in one dominant direction. Moreover, many reported common flash error mechanisms induce errors whose magnitudes (the number of error changes) are small, and independent of the alphabet size, which may be significantly larger than the typical error magnitude.

The asymmetric limited-magnitude error correcting codes can be used to speed up the writing process to flash devices (memory write is referred to as programming in the flash literature). Since the flash programming mechanism is inherently probabilistic, the introduction of "intentional" programming errors in a controlled way can significantly reduce the average programming time and improve the write performance. Such an outcome would be highly desirable given the inferiority of flash devices in write performance compared to their read performance, and to the sequential write performance of the hard-disk devices.

Asymmetric limited-magnitude error-correcting codes were proposed in [1]. The codes, proposed in that paper, were for the special case of correcting all asymmetric limited-magnitude errors within the codeword. These codes turn out to be a special case of the general construction method provided by Cassuto et al. [2].

In 2011, T. Klove and B. Bose [3] proposed systematic codes that correct single limited-magnitude systematic asymmetric errors and achieve higher rate than the ones given in [2]. They also showed how their code construction can be slightly modified to gives codes correcting symmetric errors of limited magnitude. Later T. Klove et al. [4] extended their result and gave a necessary and sufficient condition for existing a code over GF_p correcting a single asymmetric error.

As it has been already mentioned, asymmetric errors in flash memories are very common. However, there are cases in which the possible error type includes both a symmetric and an asymmetric error. For example, let us have a flash memory with n voltage levels and we should increase the voltage level of a cell with current level $t-1$ by one (which is an usual situation when programming a flash memory). In such a case the most common error we could have is overcharging the cell (increasing the level with at least 2, or to charge it less than is needed, i.e. after charging the cell stays at level $t-1$. Hence, that kind of error is a combination of the symmetric error (± 1) and the asymmetric error $(2, 3, \ldots, n)$.

The aim of this paper is to investigate the problem of finding suitable error correcting codes capable of correcting such an error. To do that we are going to use integer codes, which are designed to correct specific type of errors, in contrast to the traditional codes.

In Section 2 we are going to give some necessary notation and definitions. A new construction for integer code correcting single error of type $(\pm 1, 2)$ will be shown in Section 3. Conclusion remarks will be given in Section 4.

2 Notations and definitions

In this section we shall present notation and definitions which will be used in the next sections.

Asymmetric error correcting codes were consider first from Varshamov and Tenegolz [6] in the middle of sixties. In that work they also gave the definition of integer code. For many years these codes there were almost forgotten. The multilevel flash memory renew the interest in codes correcting asymmetric errors.

Integer codes are codes defined over finite rings of integers. A. Han Vinck and H. Morita [8] investigated integer codes with a view to magnetic recording and frame synchronization. A class of integer codes correcting specific types of errors and their application to coded modulation has been proposed by H. Kostadinov et al. [7]. Because of their flexibility integer codes are very suitable for application in multilevel flash memory. Kostadinov and Manev [5] showed a possible application of integer codes for flash memories. They have constructed integer code correcting single error of type $(1, 2)$ and gave the exact form of the check matrix.

Definition 1. *Let $\mathbb{Z}_A$ be the ring of integers modulo A. An* integer code *of length n with parity-check matrix $\mathbf{H} \in \mathbb{Z}_A^{m\times n}$, is referred to as a subset of $\mathbb{Z}_A^n$, defined by*

$$\mathcal{C}(\mathbf{H}, \mathbf{d}) = \{\mathbf{c} \in \mathbb{Z}_A^n \,|\, \mathbf{c}\mathbf{H}^T = \mathbf{d} \mod A\}$$

where $\mathbf{d} \in \mathbb{Z}_A^m$.

If $d = 0$ the code is a linear $[n, n - m]$ code over $\mathbb{Z}_A$. Without loss of generality, in this paper we shall assume that $d = 0$. We will write $\mathcal{C}(\mathbf{H})$, or only $\mathcal{C}$ of there is no possibility for ambiguity.

In this paper we consider only codes with $m = 1$ (one check symbol only). Then $\mathbf{H} = (h_1, h_2, \ldots, h_n)$, $0 \neq h_i \in \mathbb{Z}_A$ and

$$\mathcal{C}(\mathbf{H}) = \{\mathbf{c} \in \mathbb{Z}_A \,|\, \sum_{i=1}^{n} c_i h_i = 0 \mod A\}$$

The integer code is designed to correct specific type of error instead correcting number on bits in a codeword as is the case in the conventional codes. Thus, we need the following definition.

Definition 2. *Let l_j and e_i be positive integers, $j = 1, \ldots, m$, $i = 1, \ldots, s$. The code $\mathcal{C}(\mathbf{H}, d)$ is said to be a single $(l_1, l_2, \ldots, l_m, \pm e_1, \pm e_2, \ldots, \pm e_s)$-error correctable if it can correct any single error with value l_j or $\pm e_i$.*

Obviously, $\mathcal{C}(\mathbf{H}, d)$ is a single $(l_1, l_2, \ldots, l_m, \pm e_1, \pm e_2, \ldots, \pm e_s)$-error correctable code if and only if the subsets $\{h_j\, l_1, h_j\, l_2, \ldots, h_j\, l_m, \pm h_j\, e_1, \pm h_j\, e_2, \ldots, \pm\, h_j\, e_s\} \subset \mathbb{Z}_A$, are pairwise disjoint and of the same cardinality $2s + l$, for any $j = 1, 2, \ldots, n$. Thus, we have

$$A \geq (2s + l)n + 1.$$

Definition 3. *A single $(l_1, l_2, \ldots, l_m, \pm e_1, \pm e_2, \ldots, \pm e_s)$-error correctable code $\mathcal{C}(\mathbf{H}, d)$ of block length n is called perfect, when $A = (2s + l)n + 1$.*

In most of the cases perfect integer codes do not exist. We shall say that a single $(l_1, l_2, \ldots, l_m, \pm e_1, \pm e_2, \ldots, \pm e_s)$-error correctable integer code $\mathcal{C}(\mathbf{H}, d)$ of block length n over $\mathbb{Z}_A$ is *optimal* if A is the minimum value for which the code $\mathcal{C}(\mathbf{H}, d)$ exists.

3 Construction of an integer code correcting single error of type $(\pm 1, 2)$

In this section we shall investigate how to construct an integer code $\mathcal{C}(\mathbf{H})$ capable to correct a single error of type $(\pm 1, 2)$. Because the code will be single error correctable, the check matrix $\mathbf{H}$ will consist of a single row.

First, let us consider the set of integers

$$B = B(m) = \{\ 4^k l < m \mid k, l, m \in \mathbb{N},\ l \text{ is odd and } m \geq 6 \text{ is even}\}.$$

And let divide the set B into two subsets - B_0 and B_1, where

$$B_0 = \{a | 3a \equiv 0 \pmod{2m},\ \text{or } \exists\, b \in B : 2a + b \equiv 0 \pmod{2m}\} \tag{1}$$

and $B_1 = B \setminus B_0$.

Remark. For every element $a \in B(m)$ the following inequality $2a < 2m$ holds. Hence, if there exists a solution of the equation $2a+b \equiv 0 \pmod{2m}$, $a, b \in B(m)$, it is unique. Therefore, the sets B_0 and B_1 are uniquely defined.

Example 1. *Let $m = 82$. Following the definition of B, B_0 and B_1 we obtain*

$$B_m = \{4, 12, 16, 20, 28, 36, 44, 48, 52, 60, 64, 68, 76, 80\}$$

$$B_0 = \{44, 48, 52, 60, 64, 68, 72, 80\}$$

and

$$B_1 = \{4, 12, 16, 20, 28, 36\}.$$

We have the following construction for a single $(\pm 1, 2)$ error correctable integer code.

Theorem 1. *Let $m \geq 6$ is a given integer and m is even. Let us consider the sets $B(m), B_0$ and B_1. The integer code $\mathcal{C}(\mathbf{H})$ over Z_{2m} with the check matrix*

$$\mathbf{H} = (1, 3, 5, 7, \cdots, m-1 \,|\, B_1)$$

is a single $(\pm 1, 2)$ error-correctable.

Proof. The integer code $\mathcal{C}(\mathbf{H})$ is a single $(\pm 1, 2)$ error-correctable if all its syndrome values are different. Hence, to prove the theorem will be enough to show that

$$\mathbf{H} \cap (-\mathbf{H}) \cap (2\mathbf{H}) = \emptyset, \tag{2}$$

where all the operation are taken into Z_{2m}.

For convenience, let us divide $\mathbf{H}$ into 2 subsets $A_1 = (1, 3, 5, 7, \cdots, m-1)$ and B_1. So, the equation (2) is equivalent to

$$A_1 \cap (-A_1) \cap (2A_1) \cap B_1 \cap (-B_1) \cap (2B_1) = \emptyset. \tag{3}$$

One can easily see that $-A = (m+1, m+3, m+5, \cdots, 2m-1)$, and $A_1 \cap (-A_1) \cap (2A_1) = \emptyset$. Moreover,

$$A_1 \cup (-A_1) = \{2n+1 | n = 0, 1, 2, 3 \cdots, m-1\} \tag{4}$$

and

$$2A_1 = \{4n+2 | n = 0, 1, 2, 3 \cdots, m/2-1\} \tag{5}$$

On the other side, $2m$ is divisible by 4. Hence, all the elements of the sets $B_1, -B_1 = \{2m - b | b \in B_1\}$ and $2B_1$ are divisible by 4. So, using (4) and (5) we have

$$(A_1 \cup (-A_1) \cup (2A_1)) \cap (B_1 \cup (-B_1) \cup (2B_1)) = \emptyset. \tag{6}$$

The only thing that we have to show is that

$$B_1 \cup (-B_1) \cup (2B_1) = \emptyset. \tag{7}$$

It is obvious that $B_1 \cup (2B_1) = \emptyset$, because all the elements of B_1 are not divisible by 8, while all the elements of $2B_1$ are divisible by 8. We have that $B_1 \cup (-B_1) = \emptyset$, since $2m - b_i > b_j$, where $b_i, b_j \in B_1$.

To prove that $(-B_1) \cup (2B_1) = \emptyset$ we should show that $2a + b \neq 0 \pmod{2m}$, where $a, b \in B_1$. But that follows from (1) and the definition of the set B_1. Hence, using (6) and (7) we complete the proof of the theorem. $\square$

Example 2. *Let $m = 64$. For the sets B, B_0 and B_1 we have*

$$B_m = \{4, 12, 16, 20, 28, 36, 44, 48, 52, 60\}, \qquad B_0 = \emptyset,$$

$$B_1 = \{4, 12, 16, 20, 28, 36, 44, 48, 52, 60\}.$$

So, the integer code $\mathcal{C}(\mathbf{H})$ over Z_{128} with the check matrix

$$H = (1, 3, 5, 7, \cdots, 63, 4, 12, 16, 20, 28, 36, 44, 48, 52, 60)$$

is a single $(\pm 1, 2)$ error-correctable.

From the definition of B_0 we have that if $2a + b \equiv 0 \pmod{2m}$, where $a, b \in B_m$, then $a \in B_0$ and $b \in B_1$. Note that if we exchange the elements a and b, i.e., $a \in B_1, b \in B_0$, then the theorem still holds. In other words, for every such a pair (a, b), we can choose one of the elements to be in B_0 and the other one in B_1 and the above theorem still holds.

4 Conclusion

In this work we have presented a new class of single error correctable integer codes designed for an application in a flash memory. Moreover, we gave the exact form of the check matrix for those codes. The decoding complexity of the codes is linear, regarding to the code length, and can be used a look-up table to decode them. All these advantages of integer codes makes them very suitable for their usage in the practice.

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