Seventh International Workshop on Optimal Codes and Related Topics
September 6-12, 2013, Albena, Bulgaria *pp. 125-133*

About the inverse football pool problem for 9 games[1]

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Dedicated to the memory of Professor Stefan Dodunekov

Abstract. Consider the set $Q_n = \{0,1,2\}^n$ equipped with the usual Hamming distance. Denote by $T(n)$ the minimal number of spheres of radius n needed to cover Q_n. The exact values of $T(n)$ are known for $n \leq 8$. The first undecided case is $n = 9$ and it is known that $67 \leq T(9) \leq 68$. We settle the case by showing that $T(9) = 68$. The inequality $T(9) = 68$ implies $T(10) \geq 102$, $T(11) \geq 153$, $T(12) \geq 230$ and $T(13) \geq 345$ thus improving the best known lower bounds for $10 \leq n \leq 13$.

1 Introduction

In football pools one bets over 13 games. For each game he chooses between three possible outcomes – win, draw or loss. The goal is to predict correctly as many games as possible. Finding the minimal number of bets in order to guarantee certain number of correctly predicted games is known as the *football pool problem*. To put this into mathematical terms consider the set $Q_n = \{0,1,2\}^n$ with the usual Hamming distance. For $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ we define the Hamming distance $\mathbf{d}(\mathbf{x},\mathbf{y})$ as

$$\mathbf{d}(\mathbf{x},\mathbf{y}) = |\{i \mid x_i \neq y_i\}|.$$

A ball $\mathcal{B}(\mathbf{x}, r)$ and a sphere $\mathcal{S}(\mathbf{x}, r)$ with center $\mathbf{x}$ and radius r are defined as the sets

$$\mathcal{B}(\mathbf{x}, r) = \{\mathbf{y} \mid \mathbf{d}(\mathbf{x},\mathbf{y}) \leq \mathbf{r}\}, \quad \mathcal{S}(\mathbf{x},\mathbf{r}) = \{\mathbf{y} \mid \mathbf{d}(\mathbf{x},\mathbf{y}) = \mathbf{r}\}.$$

When we consider a bet as a point in Q_n the football pool problem is defined as: For $R = 1, 2$ or 3 find a set $A \subset Q_n$ of minimal cardinality such that for every $\mathbf{y} \in Q_n$ there exists $\mathbf{x} \in A$ for which $\mathbf{d}(\mathbf{x},\mathbf{y}) \leq R$. In other words we cover the space Q_n with balls of radius R. For more information on this topic the reader is referred to [2].

[1] This research is partially supported by the Bulgarian NSF under the contract I01/0003

We may consider football pools from the opposite point of view. That is, how to make certain number of bets in order to ensure that there exists a bet for which none of the games will be correctly predicted. This problem was first considered in Finish football pool magazine Veikkaaja. A subset A of Q_n is called *covering* if the spheres of radius n centered at the elements of A cover Q_n. In other words $A \subset Q_n$ is a covering if for any $\mathbf{y} \in Q_n$ there exists $\mathbf{x} \in A$ such that $\mathbf{d}(\mathbf{x}, \mathbf{y}) = n$. The minimal cardinality of a covering of Q_n is denoted by $T(n)$. Finally, a covering A of Q_n is called *optimal* if $|A| = T(n)$. The problem of finding $T(n)$ and all optimal coverings of Q_n is known as the *inverse football pool problem* (see [1]). The sequence $T(n)$ is part of the on-line encyclopedia of integer sequences [5], htpp://www.research.att.com/ njas/sequences/ number A086676

After having the exact value of $T(n)$ we are interested in finding the number of distinct optimal coverings. Two coverings A and B are equivalent if A is obtained from B by a permutation of the coordinates followed by a permutation of the elements of every coordinate. More precisely we have

Definition 1. *Two coverings A and B are equivalent if there exists a permutation $\sigma \in S_n$ and n permutations $s_1, s_2, \ldots, s_n$ of $\{0, 1, 2\}$ such that*

$$(x_1, x_2, \ldots, x_n) \in A \iff (s_1(x_{\sigma(1)}), s_2(x_{\sigma(2)}), \ldots, s_n(x_{\sigma(n)}) \in B.$$

Call a pair $(\sigma, (s_1, s_2, \ldots, s_n))$ equivalence transformation. There exist $n!6^n$ equivalence transformations and therefore, in general there exist that many copies of every covering. The full automorphism group of A consists of all equivalence transformations that map A onto itself.

The next proposition is straightforward and gives an important recursive bound on $T(n)$.

Lemma 1. *The following inequality holds $T(n) \geq \dfrac{3}{2}T(n-1)$.*

Proof. Consider an optimal covering A of Q_n, i.e. A is covering and $|A| = T(n)$. For each $i \in \{0, 1, 2\}$ denote by A_i the set of vectors $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1})$ such that $(x_1, x_2, \ldots, x_{n-1}, i) \in A$. Therefore

$$A = \{\mathbf{x}0 \mid \mathbf{x} \in A_0\} \cup \{\mathbf{x}1 \mid \mathbf{x} \in A_1\} \cup \{\mathbf{x}2 \mid \mathbf{x} \in A_2\}.$$

For $\{i, j, k\} = \{0, 1, 2\}$ it is clear that $A_i \cup A_j$ is a covering of Q^{n-1}. Therefore

$$|A_i| + |A_j| \geq T(n-1).$$

The same argument implies that $|A_i|+|A_k| \geq T(n-1)$ and $|A_j|+|A_k| \geq T(n-1)$. Summing up these inequalities gives

$$2T(n) = 2(|A_i| + |A_j| + |A_k|) \geq 3T(n-1),$$

hence $T(n) \geq \dfrac{3}{2}T(n-1)$. $\square$

Remark 1. *Suppose we know the exact value of $T(n)$ for particular n and let $T(n)$ be even. Moreover assume we know all optimal coverings of Q_n. It follows from Lemma 1 that the least feasible value of $T(n+1)$ is $T(n+1) = \frac{3}{2}T(n)$. The proof of Lemma 1 implies that for any $i, j \in \{0, 1, 2\}$, $i \neq j$ the set $A_i \cup A_j$ is a covering of Q_n and $|A_0| = |A_1| = |A_2| = \frac{1}{2}T(n)$. Therefore A_0 is contained in the intersection of two coverings of Q_n (these two coverings are $A_0 \cup A_1$ and $A_0 \cup A_2$). This observation prompts the following approach. Since $A_0 \cup A_1$ is a covering of Q_n we may consider $A = A_0 \cup A_1$ as one of the known coverings of Q_n. Go through all copies of all known coverings of Q_n and find a copy (denote this copy by B) that intersects A in at least $\frac{1}{2}T(n)$ elements. If $|A \cap B| > |A_0| = \frac{1}{2}T(n)$ then we have $|A_1 \cap A_2| \geq 1$ and therefore $|A_1 \cup A_2| < T(n)$, a contradiction.*

Thus, we have $|A \cap B| = |A_0| = \frac{1}{2}T(n)$, $A_1 = A \backslash A_0$ and $A_2 = B \backslash A_0$. What remains to be checked is whether $A_1 \cup A_2$ is a covering of Q_n. If this is the case we have a covering of Q_{n+1} with cardinality $\frac{3}{2}T(n)$. If one of the described steps (finding B and checking whether $A_1 \cup A_2$ is a covering of Q_n) fails then $T(n+1) > \frac{3}{2}T(n)$.

2 Known results

It is straightforward to show that $T(1) = 2$ and using the inequality from Lemma 1 we obtain $T(2) \geq 3$, $T(3) \geq 5$, $T(4) \geq 8$, $T(5) \geq 12$, $T(6) \geq 18$.

Using the observations from Lemma 1 and Remark 1 it is not difficult to show that $T(6) = 18$ by finding a covering of Q_6 with 18 elements. This covering is given in the following table:

1.	0 0 0 0 0 0	10.	0 2 0 2 1 1
2.	1 1 1 1 0 0	11.	0 1 1 0 2 1
3.	2 2 1 0 1 0	12.	1 0 0 1 2 1
4.	1 0 2 2 1 0	13.	2 2 0 1 0 2
5.	0 2 2 1 2 0	14.	0 1 2 2 0 2
6.	2 1 0 2 2 0	15.	1 1 0 0 1 2
7.	1 2 2 0 0 1	16.	0 0 1 1 1 2
8.	2 0 1 2 0 1	17.	2 0 2 0 2 2
9.	2 1 2 1 1 1	18.	1 2 1 2 2 2

Table 1: Optimal covering of Q_6.

Note that all optimal coverings for $n = 1, 2, 3, 4, 5$ are contained as substructures of the above covering. For example, to find a covering of Q_5 first choose any coordinate t and any two $i, j \in \{0, 1, 2\}$, $i \neq j$. Second, take all elements from the covering of Q_6 having i or j in coordinate t and then delete this coordinate. As a result we obtain optimal covering of Q_5.

Therefore we have first six exact values $T(1) = 2$, $T(2) = 3$, $T(3) = 5$, $T(4) = 8$, $T(5) = 12$, $T(6) = 18$. All these results and the bounds $T(7) \leq 29$ and $T(8) \leq 44$ are due to the Finish football pool magazine Veikkaaja. It is shown in [1] that for any $n = 1, 2, 3, 4, 5, 6$ there exists unique optimal covering of Q_n.

Observe that for all $n \leq 6$ we have $T(n) = \left\lceil \frac{3}{2} T(n-1) \right\rceil$.

The first value of n for which $T(n) \neq \left\lceil \frac{3}{2} T(n-1) \right\rceil$ is $n = 7$. It has been shown in [1] by computer search that $T(7) = 29$ while the bound from Lemma 1 implies $T(7) \geq 27$. An optimal covering of Q_7 is given in the following table. It is shown in [1] that this covering is unique.

1.	0 0 0 0 0 0 0	11.	2 2 2 2 2 2 0	21.	1 2 1 2 2 2 1
2.	1 1 1 1 0 0 0	12.	2 1 2 1 0 0 1	22.	0 1 1 0 2 0 2
3.	2 2 1 0 1 0 0	13.	0 2 0 2 0 0 1	23.	1 0 0 1 2 0 2
4.	1 0 2 2 1 0 0	14.	1 2 2 0 1 0 1	24.	0 2 2 1 2 1 2
5.	1 2 2 0 0 1 0	15.	2 0 1 2 1 0 1	25.	2 1 0 2 2 1 2
6.	2 0 1 2 0 1 0	16.	2 2 1 0 0 1 1	26.	1 1 0 0 0 2 2
7.	2 1 2 1 1 1 0	17.	1 0 2 2 0 1 1	27.	0 0 1 1 0 2 2
8.	0 2 0 2 1 1 0	18.	0 0 0 0 1 1 1	28.	2 2 0 1 1 2 2
9.	1 0 1 0 2 2 0	19.	1 1 1 1 1 1 1	29.	0 1 2 2 1 2 2
10.	0 1 0 1 2 2 0	20.	2 0 2 0 2 2 1		

Table 2: The unique optimal covering of Q_7.

Lemma 1 implies $T(8) \geq 44$ and since a covering of Q_8 with 44 elements exists we conclude that $T(8) = 44$.

The known results concerning $T(n)$ are summarized in Table 3 and are taken from [1].

n	T(n)		
1	2	7	29
2	3	8	44
3	5	9	66–68
4	8	10	99–104
5	12	11	149–172
6	18	12	224–264
		13	336–408

Table 3. Results on $T(n)$.

The upper bounds for $n = 9$ and $n = 10$ were found in [4] using so-called *tabu search*. It is shown in [3] that up to equivalence there exist two optimal coverings of Q_8. They are given in Table 4 and Table 5.

1.	0 0 0 0 0 0 0 2	16.	2 2 1 0 0 1 1 2	31.	1 2 1 2 0 0 2 0
2.	1 1 1 1 0 0 0 2	17.	1 0 2 2 0 1 1 2	32.	0 2 0 2 2 2 2 0
3.	2 2 1 0 1 0 0 1	18.	0 0 0 0 1 1 1 1	33.	2 1 2 1 2 2 2 0
4.	1 0 2 2 1 0 0 1	19.	1 1 1 1 1 1 1 1	34.	1 0 1 0 1 1 2 0
5.	1 2 2 0 0 1 0 1	20.	2 0 2 0 2 2 1 1	35.	2 2 2 2 1 1 2 0
6.	2 0 1 2 0 1 0 1	21.	1 2 1 2 2 2 1 1	36.	0 1 0 1 1 1 2 0
7.	2 1 2 1 1 1 0 2	22.	0 1 1 0 2 0 2 1	37.	1 1 0 0 2 0 1 0
8.	0 2 0 2 1 1 0 2	23.	1 0 0 1 2 0 2 1	38.	0 0 1 1 2 0 1 0
9.	1 0 1 0 2 2 0 2	24.	0 2 2 1 2 1 2 2	39.	0 1 1 0 0 2 1 0
10.	0 1 0 1 2 2 0 2	25.	2 1 0 2 2 1 2 2	40.	1 0 0 1 0 2 1 0
11.	2 2 2 2 2 2 0 2	26.	1 1 0 0 0 2 2 1	41.	2 1 0 2 1 2 0 0
12.	2 1 2 1 0 0 1 1	27.	0 0 1 1 0 2 2 1	42.	0 2 2 1 1 2 0 0
13.	0 2 0 2 0 0 1 1	28.	2 2 0 1 1 2 2 2	43.	0 1 2 2 2 1 0 0
14.	1 2 2 0 1 0 1 2	29.	0 1 2 2 1 2 2 2	44.	2 2 0 1 2 1 0 0
15.	2 0 1 2 1 0 1 2	30.	2 0 2 0 0 0 2 0		

Table 4: Optimal covering $\mathcal{A}_1$ of Q_8.

1.	0 0 0 0 0 0 0 2	16.	2 2 1 0 0 1 1 1	31.	2 2 1 2 0 0 2 0
2.	1 1 1 1 0 0 0 1	17.	1 0 2 2 0 1 1 1	32.	0 0 0 2 1 0 2 0
3.	2 2 1 0 1 0 0 2	18.	0 0 0 0 1 1 1 1	33.	0 2 0 0 0 1 2 0
4.	1 0 2 2 1 0 0 2	19.	1 1 1 1 1 1 1 2	34.	2 0 1 0 1 1 2 0
5.	1 2 2 0 0 1 0 2	20.	2 0 2 0 2 2 1 2	35.	1 2 2 2 1 1 2 0
6.	2 0 1 2 0 1 0 2	21.	1 2 1 2 2 2 1 2	36.	1 1 2 1 2 2 2 0
7.	2 1 2 1 1 1 0 1	22.	0 1 1 0 2 0 2 1	37.	2 1 0 0 2 0 1 0
8.	0 2 0 2 1 1 0 2	23.	1 0 0 1 2 0 2 2	38.	0 1 1 2 2 1 1 0
9.	1 0 1 0 2 2 0 1	24.	0 2 2 1 2 1 2 2	39.	2 0 0 1 0 2 1 0
10.	0 1 0 1 2 2 0 2	25.	2 1 0 2 2 1 2 1	40.	0 2 1 1 1 2 1 0
11.	2 2 2 2 2 2 0 1	26.	1 1 0 0 0 2 2 2	41.	0 0 2 1 2 0 0 0
12.	2 1 2 1 0 0 1 2	27.	0 0 1 1 0 2 2 1	42.	1 2 0 1 2 1 0 0
13.	0 2 0 2 0 0 1 1	28.	2 2 0 1 1 2 2 1	43.	0 1 2 0 0 2 0 0
14.	1 2 2 0 1 0 1 1	29.	0 1 2 2 1 2 2 2	44.	1 1 0 2 1 2 0 0
15.	2 0 1 2 1 0 1 1	30.	1 0 2 0 0 0 2 0		

Table 5. Optimal covering $\mathcal{A}_2$ of Q_8.

We continue by examining some properties of the two optimal coverings of Q_8. For obtaining the main result in this paper it is important to know the pairs distance distribution for both coverings. Those are given in the following table.

t	1	2	3	4	5	6	7	8
pairs from $\mathcal{A}_1$ at distance t	0	0	0	210	320	240	128	48
pairs from $\mathcal{A}_2$ at distance t	0	0	0	222	320	216	128	60

Note that in both coverings the distance between any two elements is at least 4.

The full automorphism group of $\mathcal{A}_1$ has order 384 and the full automorphism group of $\mathcal{A}_2$ has order 4.

3 Main results

The main result of this paper is given in the following Theorem.

Theorem 1. *It is true that* $T(9) \geq 68$.

Proof. Suppose there exists a covering A of Q_9 with cardinality 67. For any t, $1 \leq t \leq 9$ and any $i \in \{0, 1, 2\}$ denote by A_i^t the set of elements of A having i in coordinate number t without this coordinate. Let also $a_i^t = |A_i^t|$.

It follows from $T(8) = 44$ that for any coordinate t we have

$$\{a_0^t, a_1^t, a_2^t\} = \{21, 23, 23\} \text{ or } \{22, 22, 23\}.$$

In both cases there exists a *special element* $i \in \{0, 1, 2\}$ such that $a_j^t + a_k^t = 44$ for $\{i, j, k\} = \{0, 1, 2\}$. Note that in the case $\{21, 23, 23\}$ there exist two special elements.

Without loss of generality assume that $a_1^9 + a_2^9 = 44$. Hence, the set $A_1^9 \cup A_2^9$ is equivalent to $\mathcal{A}_1$ or $\mathcal{A}_2$.

The following Lemma provides an important property of the elements of A_1^9 and A_2^9.

Lemma 2. *Let A be a covering of Q_9 and $\mathbf{u} = (u_1, \ldots, u_8)$ and $\mathbf{v} = (v_1, \ldots, v_8)$ be two elements from $A_1^9 \cup A_2^9$ such that $\mathbf{d}(\mathbf{u}, \mathbf{v}) = 4$. If there exists t, $1 \leq t \leq 8$, such that $u_t \neq v_t$ and $\{0, 1, 2\} \setminus \{u_t, v_t\}$ is a special element for coordinate t, then $\mathbf{u}$ and $\mathbf{v}$ are not simultaneous elements of A_i^9 for $i = 1, 2$.*

Proof. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors satisfying the given properties. Denote the extensions of $\mathbf{u}$ and $\mathbf{v}$ in A without coordinate t by $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$. Since $\{0, 1, 2\} \setminus$

$\{u_t, v_t\}$ is special element for coordinate t it follows that $\overline{\mathbf{u}}$ and $\overline{\mathbf{v}}$ are elements of one of the two optimal coverings $\mathcal{A}_1$ or $\mathcal{A}_2$ of Q_8.

If $\mathbf{u}, \mathbf{v} \in A_i^t$ for $i = 1$ or 2 then $\mathbf{d}(\overline{\mathbf{u}}, \overline{\mathbf{v}}) = 3$ (since we delete coordinate t where $u_t \neq v_t$ and add coordinate 9 where $\overline{u}$ and $\overline{v}$ have one and the same element). This is a contradiction with $\mathbf{d}(\mathbf{x}, \mathbf{y}) \geq 4$ for any $\mathbf{x}, \mathbf{y} \in \mathcal{A}_i$ for $i = 1$ or 2. □

Definition 2. *A vector* $(u_1, \ldots, u_8)$ *is called characteristic vector for the covering* A *of* Q_9 *if for any* t, $1 \leq t \leq 8$ *the element* u_t *is a special element for coordinate* t.

Recall that $A_1^9 \cup A_2^9$ is equivalent to $\mathcal{A}_1$ or $\mathcal{A}_2$.

Suppose that for given characteristic vector there exist vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ from $\mathcal{A}_1$ for $i = 1$ or 2, any two of which satisfy the conditions of Lemma 2. Since at least two of them are elements of A_i^9 for $i = 1$ or 2 we have a contradiction to Lemma 2. Therefore this characteristic vector has to be rejected.

For example, consider elements 1,3,4 from covering $\mathcal{A}_1$ of Table 4. Any two of these elements satisfy the conditions of Lemma 2 for all characteristic vectors of the form

$$(u_1, u_2, u_3, u_4, 2, u_6, u_7, u_8)$$

where $u_1 = 0$ or $u_2 = 1$ or $u_3 = 0$ or $u_4 = 1$.

For all possible $3^8 = 6561$ characteristic vectors we try to find three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ from $\mathcal{A}_i$ for $i = 1$ or 2, any two of which satisfy the condition of Lemma 2.

For the first covering for any characteristic vector such a triple always exists. Hence all characteristic vectors are rejected and therefore $A_1^9 \cup A_2^9$ is not equivalent to $\mathcal{A}_1$.

For the second covering $\mathcal{A}_2$ we obtain only 4 characteristic vectors for which such a triple does not exist:

$$(00021002); (02000102); (10221020); (12200120).$$

Since the permutation (24)(56) is an automorphism of $\mathcal{A}_2$ we may consider only first and third vectors.

Thus, without loss of generality we assume that $A_1^9 \cup A_2^9 = \mathcal{A}_2$ and there are two possible characteristic vectors (00021002) and (10221020).

For a particular characteristic vector the next step is to extend each vector with 1 or 2, i.e. we have to split the elements of $\mathcal{A}_2$ into A_1^9 and A_2^9. Again we make use of Lemma 2. Let the extension of the first vector be 1, i.e. $\mathbf{x} = (00000002)$ is in A_1^9. Consider a vector $\mathbf{y} \in \mathcal{A}_2$, for which $d(\mathbf{x}, \mathbf{y}) = 4$. If there exists a coordinate for which the two entries of $\mathbf{x}$ and $\mathbf{y}$ in this coordinate and the corresponding special element are pairwise distinct (equivalently they form the set $\{0, 1, 2\}$) then it follows from Lemma 2 that $\mathbf{y}$ is extended by 2. This procedure is applicable to any vector that has already been extended. Eventually we extend all vectors. For each of the two characteristic vectors we obtain only two possible extensions. They are given in the next table (i-th element of the given vector is the extension of the i-th element of $\mathbf{A}_2$).

Characteristic vector	extensions of the elements
00021002	11122121221212112111222121112122221211222112
00021002	12122111122112112122112111222122221222112112
10221020	11222211222221111221122111122112211212212112
10221020	12222221121121111212212121212112211221122112

Up to now we know all elements of A_1^9 and A_2^9. Thus, it remains to find the elements of A_0^9, i.e. the elements with last coordinate 0. Note that $|A_0^9| = 23$, so we need 23 vectors. Let $\mathbf{xi}$ for $i = 1$ or 2 and $\mathbf{y0}$ be two elements of the covering A. If there exists a coordinate for which the two entries of $\mathbf{xi}$ and $\mathbf{y0}$ and the corresponding special element are pairwise distinct then $d(\mathbf{x}, \mathbf{y}) \geq 3$. Therefore in order to find all possible elements of A_0^9 we have to find all vectors $\mathbf{y}$ of length 8 with the described property. Direct verification shows that there exist 178 such vectors. Denote this set by $\mathcal{B}$.

Furthermore, for any two vectors $\mathbf{x}$ and $\mathbf{y}$ from A_0^9 we have that if there exists a coordinate for which both $\mathbf{x}$ and $\mathbf{y}$ differ from the corresponding special element then $\mathbf{d}(\mathbf{x}, \mathbf{y}) \geq 4$. Therefore we have to find a subset A_0^9 of $\mathcal{B}$ with cardinality 23 any two elements of which satisfy the above property. In addition, for any coordinate t, $1 \leq t \leq 9$ of the set $A = A_0^9 \cup A_1^9 \cup A_2^9$ we must have $\{a_0^t, a_1^t, a_2^t\} = \{23, 23, 21\}$ or $\{23, 22, 22\}$.

Computer search finds no such set.

Therefore $T(9) > 67$ and since there exists a covering of Q_9 with 68 elements we conclude that $T(9) = 68$. □

In order to verify the computer search results, all the computations have been carried out independently by different programs written on Pascal and C++ developed by the authors. The time needed to perform the steps of the computation ranges from a few minutes to a few hours for the last step.

The exact value $T(9) = 68$ and Lemma 1 imply that

$$T(10) \geq 102, T(11) \geq 153, T(12) \geq 230, T(13) \geq 345.$$

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