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A note on the existence of spreads in projective Hjelmslev spaces

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Dedicated to the memory of Professor Stefan Dodunekov

Abstract. We provide new parameters for spreads for which the standard combinatorial necessary condition is not sufficient.

Let R be a finite chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$. It is well-known that every finite module ${}_RM$ over R is isomorphic to a direct sum of cyclic modules, i.e.

$$ {}_RM \cong \bigoplus_{i=1}^{k} R/(\operatorname{Rad} R)^{\lambda_i}, $$

for some uniquely determined partition $\lambda = (\lambda_1, \ldots, \lambda_k) \vdash \log_q|M|$. Here $m \geq \lambda_1 \geq \ldots \geq \lambda_k > 0$. The integer k is called the rank, and the partition λ – the shape of M. The following counting formula gives the number of submodules of fixed shape μ contained in a module of shape λ [1, 9].

Theorem 1. Let ${}_RM$ be a module of shape λ. For every partition μ satisfying $\mu \leq \lambda$ the module ${}_RM$ has exactly

$$ \begin{bmatrix} \lambda \\ \mu \end{bmatrix}_q := \prod_{i=1}^{\infty} q^{\mu'_{i+1}(\lambda'_i - \mu'_i)} \cdot \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \mu'_i - \mu'_{i+1} \end{bmatrix}_q \tag{1} $$

submodules of shape μ. In particular, the number of free rank s submodules of ${}_RM$ equals

$$ q^{s(\lambda'_1 - s) + \cdots + s(\lambda'_{m-1} - s)} \cdot \begin{bmatrix} \lambda'_m \\ s \end{bmatrix}_q . $$

Here $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ is the conjugate partition to λ.

Set $M = {}_RR^n$ and $M^* = M \setminus \theta M$, where θ is any generator of $\operatorname{Rad} R$. Let $\mathcal{P} = \{Rx \mid x \in M^*\}$, and $\mathcal{L} = \{Rx + Ry \mid x, y \text{ linearly independent}\}$ be called the set of points and the set of lines, respectively, with incidence I given by set-theoretical inclusion. Two points Rx and Ry are called neighbours if $Rx \cap Ry \leq \theta M$. Two lines K and L are i-th neighbours if for every point x on K there is a point y on L which is an neighbour to x, and conversely, for every point y on L there is a point x on K which is an neighbour to y. The relation neighborhood is an equivalence relation on $\mathcal{P}$, as well as on $\mathcal{L}$. The incidence structure $(\mathcal{P}, \mathcal{L}, I)$, together with the neighbourhood relations defined above, is called a left projective Hjelmslev space over the chain ring R and is denoted by $\mathrm{PHG}({}_RR^n)$.

A set of points H in $\Pi = \mathrm{PHG}({}_RR^n)$ is called a Hjelmslev subspace if for any two points $x, y \in H$ there is at least one line incident with x and y which is entirely contained in H. Equivalently, the pointset H is an Hjelmslev subspace if it contains all free rank 1 submodules in a free submodule of ${}_RR^n$. The intersection of Hjelmslev subspaces is not necessarily a Hjelmslev subspace. A nonempty set of points S in Π is called a subspace if it contains all points (free rank 1 submodules) in any (not necessarily free) submodule of ${}_RR^n$. A subspace consisting of the points in a submodule of type λ is called a subspace of type λ. The intersection of two subspaces of ${}_RR^n$ is again a subspace. The neighbor relations defined above can be extended to any two subspaces of the same type in an obvious way.

In order to save space we refer to [2, 4, 6, 7, 10, 11] for a more detailed introduction to finite chain rings, modules over finite chain rings, and projective spaces over such rings.

Let $\Pi = \mathrm{PHG}({}_RR^n)$ and let $\lambda = \lambda_1, \ldots, \lambda_k)$, $k \leq n$, be a k tuple of integers with $m \geq \lambda_1 \geq \ldots \geq \lambda_k > 0$. For the sake of convenience we set $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_{k+1} = \ldots = \lambda_n = 0$. A λ-spread of Π is a set $\mathcal{S}$ of subspaces of type λ in $\mathrm{PHG}({}_RR^n)$ which form a partition of the pointset of Π. It is obvious that if a λ-spread exists the number of points in a subspace of type λ should divide the number of points in Π, i.e. $\begin{bmatrix}\lambda \\ \boldsymbol{m}_1\end{bmatrix}_{q^m}$ divides $\begin{bmatrix}\boldsymbol{m}_n \\ \boldsymbol{m}_1\end{bmatrix}_{q^m}$. It is known that for $\lambda = \boldsymbol{m}_k$ this condition is also sufficient, in full analogy with the classical theorem on spreads in finite projective spaces (cf. [3]). In other words, a spread of $(k-1)$-dimensional Hjelmslev subspaces exists if and only if k divides n [5, 8].

It has been pointed out in [5] that this necessary condition is not always sufficient.

Theorem 2. Let R be a chain ring of nilpotency index 2. Let $n \geq 4$ be even and $\Pi = \mathrm{PHG}({}_RR^n)$. There exists no λ-spread of $\Pi = \mathrm{PHG}({}_RR^n)$ with $\lambda = (\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2-1},0)$.

In this note we generalize this result by extending the set of the shapes for which the necessary condition is not sufficient.

Theorem 3. Let R be a finite chain ring of nilpotency index 2 and let $\Pi = \mathrm{PHG}({}_RR^n)$ be the corresponding (left) projective Hjelmslev space. There exists no λ-spread of $\Pi = \mathrm{PHG}({}_RR^n)$ with $\lambda = (\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2-a},\underbrace{0,\ldots,0}_{a})$, where $1 \leq a \leq \frac{n}{2}-1$.

First we state without proof two easy lemmas which will be used in the proof of our main theorem.

Lemma 4. Let $\Pi = \mathrm{PHG}({}_RR^n)$ and let x be a point in Π. For a subspace U of codimension a, $1 \leq a \leq n/2-1$, and a subspace S of type $\lambda = (\underbrace{2,\ldots,2}_{n/2},\underbrace{1,\ldots,1}_{n/2-a},\underbrace{0,\ldots,0}_{a})$, $[x]\cap S \subseteq [x]\cap U$ implies $S \subseteq [U]$.

Lemma 5. Let $\Pi = \mathrm{PHG}({}_RR^n)$ and let x be a point in Π. Then the nonemty sets $[x]\cap S$, $S \in \mathcal{S}$, form a parallel class of affine subspaces of codimension a in $[x] \cong \mathrm{AG}(n-1,q)$.

Proof. (Theorem 3) Assume for a contradiction that such a spread $\mathcal{S}$ does exist. Let us count in two different ways the number of pairs $(S,[U])$, where $S \in \mathcal{S}$, $[U]$ is a neighbour class of Hjelmslev subspaces of codimension a and $S \subset [U]$ (as sets of points). There are $|\mathcal{S}|$ possibilities for the subspace S where

$$|\mathcal{S}| = \frac{q^{n-1}\begin{bmatrix} n \\ 1 \end{bmatrix}_q}{q^{n-2}\begin{bmatrix} n/2 \\ 1 \end{bmatrix}_q} = q(q^{n/2}+1).$$

On the other hand, the number of neighbour classes of Hjelmslev subspaces of codimension a containing a fixed subspace S of type λ is equal to $\begin{bmatrix} n/2 \\ a \end{bmatrix}_q$ (cf. Theorem 1). Therefore the number of pairs $(S,[U])$ with $S \subset [U]$ is equal to

$$q\cdot(q^{n/2}+1)\begin{bmatrix} n/2 \\ a \end{bmatrix}_q. \qquad (2)$$

Let $x \cap S \cap [U]$, $S \in \mathcal{S}$, $S \cap [U] \neq \varnothing$, be an arbitrary point. The incidence structure formed by the points $[x] \cap V \neq \varnothing$, $V \in [U]$, and hyperplanes the subspaces from $[U]$, is isomorphic to $\mathrm{PG}(n-1, q)$ from which a subspace of (projective) dimension $a-1$ is deleted. We denote this geometry by Δ_U. Denote the missing part by Z_∞, The points of Δ contained in S form a subspace of dimension $n/2 - 1$. Clearly $\langle Z_\infty, S\rangle$ is a $(n/2 + a - 1)$-dimensional subspace of Δ. Consider another subspace $T \in \mathcal{S}$.

We shall estimate the number of point classes $[x]$ such that $[x] \cap S \neq \varnothing$, and $[y] \cap T \neq \varnothing$. We claim that the number of such point classes $[x]$ is at most $(q^a - 1)/(q - 1)$. Otherwise, there exist classes $[x]$ and $[y]$ such that

$$[x] \cap S \neq \varnothing, [y] \cap S \neq \varnothing, [x] \cap T \neq \varnothing, [y] \cap T \neq \varnothing,$$

and such that the lines

$$\langle [x] \cap S, [x] \cap T\rangle, \text{ and } \langle [y] \cap S, [y] \cap T\rangle$$

meet Z_∞ in the same point. Then

$$\langle [x] \cap S, [y] \cap S\rangle, \text{ and } \langle [x] \cap T, [y] \cap T\rangle$$

are coplanar and hence meet outside Z_∞. This contradicts the fact that S and T belong to a spread.

Now the number of λ-subspaces contained in the class $[U]$ which has at least one λ-subspace from the spread $\mathcal{S}$, S say. Let $[x]$ be a neighbor class of points with $[x] \cap S \neq \varnothing$. By Lemma 5 each of the $q^a - 1$ segments parallel to $[x] \cap S$ must lie in some subspace from the spread. Moreover each subspace contains no more than $(q^a - 1)/(q - 1)$ segments. Hence the number of λ-subspaces in $[U]$ is at least

$$1 + (q^a - 1)\frac{q^{n/2} - 1}{q - 1} \cdot \frac{q - 1}{q^a - 1} = q^{n/2}.$$

This is a partial spread of Δ_U with exactly one subspace missing. The missing part contains Z_∞. So, the number of λ-subspaces from $\mathcal{S}$ in $[U]$ is exactly $q^{n/2}$ and hence a neighbour class $[U]$ of subspaces of codimension a contains either $q^{n/2}$ or 0 subspaces from $\mathcal{S}$. It follows that $q^{n/2}$ divides $q \cdot (q^{n/2} + 1)\begin{bmatrix} n/2 \\ a \end{bmatrix}_q$, a contradiction to $n \geq 4$. □

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References

[1] G. Birkhoff, Subgroups of abelian groups, Proc. of The London Math. Society 38(2)(1934/35), 385–401.

[2] W. E. Clark, D. A. Drake, Finite chain rings, Abh. Math. Sem. der Univ. Hamburg 39(1974), 147–153.

[3] J. W. P. Hirschfeld, Finite Projective Geometries in Three Dimensions, Clarendon Press, Oxford, 1985.

[4] Th. Honold, I. Landjev, Linear Codes over Finite Chain Rings and Projective Hjelmslev Geometries, in: Codes over Rings (ed. P. Solé), World Scientific, 2009, 60–123.

[5] M. Kiermaier, I. Landjev, Designs in projective Hjelmslev spaces, in: Contemporary Mathematics vol. 579, Theory and Applications of Finite Fields (eds. M. Lavrauw et al.), AMS, 2012, 111–122.

erasures in random network coding, IEEE Trans. Inf. Th. 54(2008), 3579–3591.

[6] A. Kreuzer, Hjelmslev-Räume, Resultate der Mathematik 12(1987), 148–156.

[7] A. Kreuzer, Projektive Hjelmslev-Räume, Dissertation, technische Universität München, 1988.

[8] I. Landjev, Spreads in Projective Hjelmslev Geometries, Lect. Note in Comp. Science 5527(2009), 186–194.

[9] I. G. MacDonald, Symmetric Functions and Hall Polynomilas, Oxford University Press, 2nd edition, 1995.

[10] B. R. McDonald, Finite rings with Identity, Marcel Dekker, New york, 1974.

[11] A. A. Nechaev, Finite principal ideal rings, Russian Acad. of Sciences, Sbornik Mathematics 2091973), 364–382.