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New good (n, r)-arcs in PG(2, 29)[1]

Rumen Daskalov daskalovrn@gmail.com
Department of Mathematics, Technical University of Gabrovo,
5300 Gabrovo, BULGARIA
Elena Metodieva metodieva@tugab.bg
Department of Mathematics, Technical University of Gabrovo,
5300 Gabrovo, BULGARIA

Dedicated to the memory of Professor Stefan Dodunekov

Abstract. An (n, r)-arc is a set of n points of a projective plane such that some r, but no $r + 1$ of them, are collinear. The maximum size of an (n, r)-arc in PG(2, q) is denoted by $m_r(2, q)$. In this paper we establish that $m_{11}(2, 29) \geq 258$, $m_{13}(2, 29) \geq 325$, $m_{14}(2, 29) \geq 361$, $m_{17}(2, 29) \geq 452$ and $m_{18}(2, 29) \geq 474$. The presented results improve the respective lower bounds in [9].

1 Introduction

Let $\mathrm{GF}(q)$ denote the Galois field of q elements and $\mathrm{V}(3, q)$ be the vector space of row vectors of length three with entries in $\mathrm{GF}(q)$. Let $\mathrm{PG}(2, q)$ be the corresponding projective plane. The *points* (x_1, x_2, x_3) of $\mathrm{PG}(2, q)$ are the 1-dimensional subspaces of $\mathrm{V}(3, q)$. Subspaces of dimension two are called *lines*. The number of points and the number of lines in $\mathrm{PG}(2, q)$ is $q^2 + q + 1$. There are $q + 1$ points on every line and $q + 1$ lines through every point.

Definition 1. *An (n, r)-arc is a set of n points of a projective plane such that some r, but no $r + 1$ of them, are collinear.*

Definition 2. *An (l, t)-blocking set S in $\mathrm{PG}(2, q)$ is a set of l points such that every line of $\mathrm{PG}(2, q)$ intersects S in at least t points, and there is a line intersecting S in exactly t points.*

An (n, r)-arc is the complement of a $(q^2 + q + 1 - n, q + 1 - r)$-blocking set in a projective plane and conversely.

Definition 3. *Let M be a set of points in any plane. An i-secant is a line meeting M in exactly i points. Define τ_i as the number of i-secants to a set M.*

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In terms of τ_i the definitions of an (n,r)-arc and an (l,t)-blocking set become the following: An (n,r)-arc is a set of n points of a projective plane for which $\tau_i \geq 0$ for $i < r$, $\tau_r > 0$ and $\tau_i = 0$ when $i > r$. An (l,t)-blocking set is a set of l points of a projective plane for which $\tau_i = 0$ for $i < t$, $\tau_t > 0$ and $\tau_i \geq 0$ when $i > t$.

A survey of (n,r)-arcs with the best known results was presented in [8]. After this publication many improvements were obtained in [4], [5] and [3]. Summarizing these improvements, Ball and Hirschfeld [2] presented a new table with bounds on $m_r(2,q)$ for $q \leq 19$. It follows from these tables that the exact values of $m_r(2,q)$ are known only for $q \leq 9$. Some new improvements were made in recent years. A (79,6) arc in PG(2,17) and a (126,8) arc in PG(2,19) are given in [9]. A $(95,7)$-arc, a $(183,12)$-arc, a $(205,13)$-arc in PG(2,17) and a $(243,14)$-arc and a $(264,15)$-arc n PG(2,19) have been presented in [6]. A $(265,15)$-arc in PG(2,19) can be found in [9]. Gulliver constructed an optimal (78,8) arc in PG(2,11) (see [1]). A table for $m_r(2,q)$, $q \leq 19$, is maintained by S. Ball [1]. A $(286,16)$-arc in PG(2,19), new results and tables with lower and upper bounds on $m_r(2,q)$ for $q = 23, 25, 27$ are presented in [6] and [7].

Our approach to obtaining good (n,r)-arcs is a local computer search. The neighborhood structure is a simple one. Given an arc, then its neighborhood consists of all arcs that can be obtained from the given arc by adding new points or deleting some points. The choice of a start solution is based on some heuristic observations. The cost function is chosen to favor as local optima arcs with a small number of r-secants. The computation time is in the order of several minutes up to a few hours on a PC. Similar techniques are employed for the construction of (l,t)-blocking sets.

In order to present the results in more concise form, the points in PG(2,29) are in lexicographic order and each point is associated with its number.

Some points in PG(2,29)

Number	Point	Number	Point	Number	Point
1	(0,0,1)	...	...	610	(1,19,28)
2	(0,1,0)	300	(1,9,8)	...	...
3	(0,1,1)	301	(1,9,9)	869	(1,28,26)
4	(0,1,2)	302	(1,9,10	870	(1,28,27)
5	(0,1,3)	...	...	871	(1,28,28)

The following lower bounds on $m_r(2,29)$ are given in [9]. The arcs in bold are optimal.

Lower bounds on $m_r(2, 29)$ in [9]

r	2	3	4	5	6	7	8	9	10
$m_r(2,29)$	**30**	43	70	94	126	146	181	201	233
r	11	12	13	14	15	16	17	18	19
$m_r(2,29)$	255	291	324	342	**407**	**436**	451	471	499
r	20	21	22	23	24	25	26	27	28
$m_r(2,29)$	521	563	580	619	658	683	715	755	784

In this paper we improve some of these results ($r = 11, 13, 14, 17, 18$) by constructing five new arcs.

2 The new arcs in PG(2, 29)

Theorem 1. *There exist a* $(258, 11)$*-arc, a* $(325, 13)$*-arc, a* $(361, 14)$*-arc, a* $(452, 17)$*-arc and a* $(474, 18)$*-arc in* PG(2,29).

Proof. 1. The set of points having numbers 64 65 79 83 84 86 88 89 90 92 93 102 105 110 117 118 124 131 134 135 136 138 141 143 144 153 155 156 157 162 164 166 168 170 174 179 180 182 184 186 187 194 195 201 202 207 208 210 212 215 224 226 227 229 231 235 236 239 241 250 257 258 262 271 272 273 276 279 289 290 295 301 303 304 306 307 309 310 318 320 323 325 327 332 335 339 344 346 347 348 349 350 351 352 356 357 358 362 367 371 375 377 379 382 384 385 390 397 400 401 402 403 405 409 410 416 419 421 424 425 426 429 435 437 438 442 443 447 448 455 457 460 464 465 466 468 470 474 481 483 487 491 499 500 506 507 508 511 513 520 523 524 525 526 527 531 537 540 546 550 551 553 554 558 565 570 572 577 579 582 587 588 592 594 595 599 601 604 605 606 612 616 617 618 619 624 627 628 632 633 639 640 644 648 651 654 660 661 662 664 665 671 672 674 680 681 682 684 685 686 687 693 700 703 705 707 714 718 720 722 723 724 733 734 737 740 742 747 749 750 752 757 760 762 764 765 771 777 779 780 786 792 795 801 803 804 806 813 816 818 825 833 839 846 847 853 862 868 forms a (258,11)-arc $\mathcal{K}_1$ in PG(2,29) with secant distribution

$$\tau_0 = 29,\ \tau_1 = 5,\ \tau_2 = 2,\ \tau_3 = 3,\ \tau_4 = 8,\ \tau_5 = 26,$$

$$\tau_6 = 24,\ \tau_7 = 76,\ \tau_8 = 95,\ \tau_9 = 143,\ \tau_{10} = 223,\ \tau_{11} = 237$$

2. Deleting from $\mathcal{K}_1$ the points with numbers 79 102 110 141 179 195 201 295 303 349 448 506 511 513 527 540 546 582 587 616 627 648 671 703 734 825 and adding the points having numbers 67 70 94 97 108 113 114 115 122 129 130

149 167 190 197 199 213 216 232 240 247 252 256 261 263 266 282 283 284 293 300 312 321 372 373 387 420 427 436 439 456 461 471 478 479 480 493 504 512 519 533 538 563 567 568 573 581 596 598 620 623 634 647 649 658 683 695 696 702 711 725 731 736 746 761 770 781 784 793 796 798 800 807 821 823 824 826 831 832 834 836 841 857 we obtain a (325,13)-arc $\mathcal{K}_2$ in PG(2,29) with secant distribution

$$\tau_0 = 31,\ \tau_4 = 2,\ \tau_5 = 2,\ \tau_6 = 4,\ \tau_7 = 9,\ \tau_8 = 21,$$

$$\tau_9 = 33,\ \tau_{10} = 81,\ \tau_{11} = 172,\ \tau_{12} = 230,\ \tau_{13} = 286$$

3. From $\mathcal{K}_2$ by deleting points with numbers 538 620 649 824 and adding points with numbers 66 79 102 110 121 141 195 245 265 269 295 303 313 324 336 344 378 396 397 448 490 506 511 513 527 556 582 587 627 631 648 671 703 734 743 799 814 825 850 870 we obtain a (361,14)-arc in PG(2,29). The secant distribution of this arc is

$$\tau_0 = 31,\ \tau_8 = 2,\ \tau_9 = 14,\ \tau_{10} = 26,\ \tau_{11} = 54,\ \tau_{12} = 142,\ \tau_{13} = 298,\ \tau_{14} = 304$$

4. The set of points having numbers 1 2 19 20 21 22 23 24 25 27 28 29 30 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 50 51 52 53 54 55 56 57 58 59 60 62 68 69 71 73 74 75 76 78 80 81 87 91 95 96 98 100 101 103 104 106 107 109 111 112 116 119 120 123 125 126 128 132 133 137 139 140 142 145 146 147 148 150 151 154 158 160 163 165 169 172 173 175 176 177 178 181 183 185 188 193 196 198 200 203 204 205 206 209 211 217 218 219 220 221 222 228 230 233 234 237 238 242 243 244 246 251 253 254 255 259 260 264 267 268 270 274 275 277 278 280 281 285 287 288 291 292 294 296 297 298 302 305 308 311 315 316 317 319 326 328 329 330 331 333 334 337 338 340 341 342 343 345 353 355 359 360 361 363 364 365 366 368 369 370 374 376 380 381 383 388 389 392 393 394 395 398 399 404 406 407 408 411 413 414 415 417 422 423 428 430 431 432 434 440 441 444 445 449 450 451 452 453 454 458 459 462 463 467 469 472 473 475 476 477 484 485 486 488 489 492 494 495 497 498 501 502 503 509 510 516 517 518 521 522 528 529 530 532 534 535 536 541 542 543 545 547 548 549 555 557 559 560 561 564 566 569 571 574 575 576 578 580 583 584 585 586 590 591 593 600 602 603 607 608 609 610 611 613 615 616 621 622 625 626 629 630 635 636 638 641 642 643 646 650 652 653 656 657 659 663 666 667 668 669 670 673 676 677 678 679 688 689 690 691 694 697 698 699 704 708 709 710 712 713 715 716 717 721 726 727 728 729 730 732 735 739 744 748 751 753 754 755 756 759 763 766 767 768 769 772 773 774 775 778 782 787 788 789 790 791 794 797 802 805 808 809 810 811 812 815 817 819 820 822 827 828 829 830 835 837 838 840 842 843 848 849 851 852 854 856 859 861 863 864 866 867 forms a (419,13)-blocking set in PG(2,29) with secant distribution

$$\tau_{13} = 239,\ \tau_{14} = 406\ \tau_{15} = 197,\ \tau_{28} = 17,\ \tau_{29} = 12$$

The complement of this blocking set is a $(452, 17)$-arc in PG(2, 29).

5. The set of points having numbers 7 10 19 21 22 23 24 25 27 28 29 30 32 33 34 35 36 37 38 39 40 41 42 43 44 45 46 47 48 49 51 52 54 55 56 57 58 59 60 62 68 69 71 73 74 75 78 80 81 87 91 96 98 100 101 104 106 107 109 111 112 116 119 120 123 125 126 128 133 137 139 140 142 145 146 147 148 150 151 154 158 160 163 165 172 173 175 176 178 181 183 185 188 193 196 198 200 203 204 205 206 209 211 217 218 219 220 221 222 228 230 233 234 237 238 242 243 244 246 253 254 255 259 260 264 267 268 270 274 275 277 278 280 281 285 287 291 292 294 297 298 302 305 308 311 315 316 317 319 326 328 329 330 333 334 337 338 340 341 342 343 345 353 355 359 360 361 363 364 365 366 368 369 374 376 380 381 383 388 389 392 393 394 395 398 399 404 406 407 408 411 413 414 415 417 422 423 428 430 431 432 434 440 441 444 445 449 450 451 453 454 458 459 462 463 467 469 472 473 475 476 477 484 485 486 488 492 494 495 497 498 501 502 503 509 510 516 517 518 521 522 523 529 530 532 535 536 541 542 543 545 547 548 549 555 557 559 560 561 564 566 569 571 574 575 576 578 580 583 584 585 586 590 591 593 600 602 603 607 609 610 611 613 615 621 622 625 626 629 630 635 636 638 641 642 643 646 650 652 653 656 657 659 663 666 667 668 669 670 673 676 677 678 679 688 689 690 691 694 697 698 699 704 708 709 710 712 713 715 716 717 721 726 727 728 729 730 732 739 744 748 751 753 754 755 756 759 763 766 767 768 769 772 773 774 775 778 782 787 788 789 790 791 794 797 802 805 808 810 811 812 815 817 819 820 822 827 828 829 830 835 837 838 840 842 843 848 849 851 852 856 859 861 863 864 866 867 forms a (397,12)-blocking set in PG(2,29) with secant distribution

$$\tau_{12} = 214, \tau_{13} = 323,\ \tau_{14} = 218,\ \tau_{15} = 85,\ \tau_{16} = 3,$$

$$\tau_{25} = 1,\ \tau_{26} = 4,\ \tau_{27} = 11,\ \tau_{28} = 6,,\ \tau_{29} = 6$$

The complement of this blocking set is a $(474, 18)$-arc in PG(2, 29).

□

The new lower bounds on $m_r(2, 29)$

r	2	3	4	5	6	7	8	9	10
$m_r(2,29)$	**30**	43	70	94	126	146	181	201	233
r	11	12	13	14	15	16	17	18	19
$m_r(2,29)$	*258*	291	*325*	*361*	**407**	**436**	*452*	*474*	499
r	20	21	22	23	24	25	26	27	28
$m_r(2,29)$	521	563	580	619	658	683	715	755	784

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