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# On network codes and partial spreads

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**Dedicated to the memory of Professor Stefan Dodunekov**

By $\mathcal{G}_q(n,l)$ we denote the set of all $l$-dimensional subspaces of a fixed $n$-dimensional vector space, say $W$, over the finite field $K = \mathbb{F}_q$. Sometimes this space is also referred to as a Grassmannian. The set $\mathcal{G}_q(n,l)$ becomes a metric space by

$$\mathrm{d}(U,V) = \dim U + \dim V - 2\dim(U \cap V) = 2l - 2\dim(U \cap V)$$

for $U, V \in \mathcal{G}_q(n,l)$. A subset $\mathcal{C}$ of $\mathcal{G}_q(n,l)$ is called a network code, more precisely, a network code of constant dimension. As usual we denote by

$$\mathrm{d}(\mathcal{C}) = \min\{\mathrm{d}(U,V) \mid U, V \in \mathcal{C},\, U \neq V\} \in 2\mathbb{N}$$

the minimum distance of $\mathcal{C}$ for $|C| > 1$. In case $|\mathcal{C}| = 1$ we put $\mathrm{d}(\mathcal{C}) = 0$. In network coding one is interested in codes $\mathcal{C} \subseteq \mathcal{G}_q(n,l)$ of large size where the minimum distance $d$ is given. Thus we define

$$\mathcal{A}_q(n,d,l) = \max\{|\mathcal{C}| \mid \mathcal{C} \subseteq \mathcal{G}_q(n,l),\, \mathrm{d}(\mathcal{C}) \geq d\}.$$

Computing $\mathcal{A}_q(n,d,l)$ is a notoriously hard problem, and only for particular parameters $q, n, d$ and $l$ the exact value is known. In this note we are looking at codes $\mathcal{C}$ in the special case $d = 2l$, i.e., the elements of $\mathcal{C}$ form a partial spread. In other words, any two different elements of $\mathcal{C} \subseteq \mathcal{G}_q(n,l)$ intersect only in the zero space. A partial spread is called a spread if the elements of $\mathcal{C}$ cover the full space $K^n$. Thus $\mathcal{A}_q(n,2l,l)$ is the largest size of a partial spread (resp. a spread) in $K^n$ consisting of $l$-dimensional subspaces.

Let $n = kl + r$ where $k, l \in \mathbb{N}$ and $0 \leq r < l$. If $r = 0$ then it is well-known that a spread of $l$-dimensional subspaces in $K^n$ exists. This trivially implies that

$$\mathcal{A}_q(n,2l,l) = \frac{q^n - 1}{q^l - 1}.$$

For $r = 1$ we have the following result.

**Theorem.** If $n = kl + 1 \geq 2l + 1$ where $k, l \in \mathbb{N}$ then

$$\mathcal{A}_q(n, 2l, l) = q^{n-l} + q^{n-2l} + \cdots + q^{n-(k-1)l} + 1 = \left\lfloor \frac{q^n - 1}{q^l - 1} \right\rfloor - (q - 1).$$

Unfortunately, we do not know the exact value of $\mathcal{A}_q(n, 2l, l)$ if $r > 1$. The counting argument used to prove the Theorem does not work for $r > 1$. However, based on the Theorem above and a result of EL-Zanati, Jordon, Seelinger, Sissokho and Spence [2] in which $n = 3k + 2 \geq 8$ (hence $r = 2$) we may ask the

**Question.** Do we always have

$$\begin{aligned} \mathcal{A}_q(n, 2l, l) &= q^{n-l} + q^{n-2l} + \cdots + q^{n-(k-1)l} + q^{r-1} \\ &= \left\lfloor \tfrac{q^n - 1}{q^l - 1} \right\rfloor - q^{r-1}(q - 1) \end{aligned}$$

if $n = kl + r$ where $k \geq 2$ and $0 < r < l$?

**Remark.** The Theorem above answers a question posed by Bu in [1] positively for $r = 1$. However, [2] shows that the answer is negative for $r > 1$.

We are deeply indebted to Oksana Yakimova. From her we learned the counting arguments to prove the Theorem.

The proof will appear in a forthcoming paper in which we discuss the function $\mathcal{A}_q(n, d, l)$ in more detail.

## References

[1] T. Bu, Partitions of a vector space, *Discr. Math.* **31** (1980), 79-83.

[2] S. El-Zanati, H. Jordon, G. Seelinger, P. Sissokho, L. Spence, The maximum size of a partial 3-spread in a finite vector space over GF(2), *Des. Codes Cryptogr.* **54** (2010), 101-107.