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On (4,9,96) binary orthogonal arrays

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Dedicated to the memory of Professor Stefan Dodunekov

Abstract. We consider the points of weight 3 on the putative binary orthogonal arrays of strength 4, length 9 and cardinality 96 of minimum distance 1. We obtain some restrictions which reduce the possibilities for the structure of such arrays.

1 Introduction

For basic definitions and results on orthogonal arrays (or designs in Hamming spaces) we refer to [5, 6]. More specific results and approaches are given in [2]. In this note we continue investigations from [2] by adding certain new necessary conditions. We conjecture that any $(4, 9, 96)$ binary orthogonal array (if it exists) must have minimum distance of 2.

2 Overview of the method of investigation

Let C be a BOA of parameters (strength,length,cardinality) $= (4, 9, 96)$ whose existence is mentioned as undecided in Table 12.1 from the book [5]. In [2] we described some advance on the investigation od such BOA. We start with calculation of all feasible distance distributions and then reduce the possibilities by certain algorithms, called A and B in [2], and some ad hoc arguments.

In particular, Algorithm A gives all possible types of columns in the targeted BOA by calculation of vectors $(x_1^{(s)}, x_2^{(s)}, \dots, x_n^{(s)})$ and $(y_0^{(s)}, y_1^{(s)}, \dots, y_{n-1}^{(s)})$

where $x_i^{(s)}$ and $y_i^{(s)}$ are the numbers of 1's and 0's, respectively, in the intersection of that column (the notation s is for the column) and the i-block (the i-block, $i \in \{0, 1, \dots, n\}$, as the set of all rows of weight i in the matrix of our BOA). Algorithm B is applied for investigation of the relations between C and its relatives derived by simultaneous cut of several columns (defined by the support of some point $t \in C$). In particular, Algorithm B gives the numbers $p_{i,j}(t)$, equal to the number of the points of C of weight j and at distance i from t. Furthermore, in every specific situation we are able to calculate the numbers $p_{i,j}^{\varepsilon,s}(t)$, $\varepsilon = 0, 1$, equal to the number of points of C of weight j and at distance i from t which are intersected by the s-th column in the symbol ε.

Assume that the minimum distance of C is 1. The investigation in [2] in this case leave only two possible distance distributions for any two points at distance 1. Here we complete this case therefore proving that the minimum distance of C is 2. The results in [2] come, in some sense, from considerations of the i-blocks for $i = 0, 1, 2$. Here we add further arguments to investigate the 3-block of C.

3 Investigations of the points of weight 3

Let $t \in C$ be a point of weight 3 and (i_1, i_2, i_3) be its support. We know the types of the columns i_1, i_2 and i_3 and all possible distance distributions of t from Algorithm B with $\tau_0 = 3$ (more precisely, we know the numbers $p_{i,j}(t)$).

Theorem 1. *The identities*

$$x_i^{(i_1)} + x_i^{(i_2)} + x_i^{(i_3)} = p_{i+1,i}(t) + 2p_{i-1,i}(t) + 3p_{i-3,i}(t)$$

hold true for every $i = 3, 4, \dots, 9$.

Proof. Denote by N the numbers of the 1's in the intersection of the i-block and the columns i_1, i_2 and i_3. Then clearly $N = x_i^{(i_1)} + x_i^{(i_2)} + x_i^{(i_3)}$.

Let u be an arbitrary point from the i-block of C. Then we have $wt(t * u) \in \{0, 1, 2, 3\}$ for every u from the i-block. Therefore out count includes exactly three terms: one 1 for every u at distance $i + 1$ from t (there are $p_{i+1,i}(t)$ such points), two 1's for every u at distance $i - 1$ from t and three 1's for every u at distance $i - 3$ from t which results in $p_{i+1,i}(t) + 2p_{i-1,i}(t) + 3p_{i-3,i}(t) = N$. □

For fixed point $t \in C$ denote by $p_{i,j}^{(1,s)}(t)$ the number of the points of C of weight j (i.e. from the j-block) at distance i from t whose intersection with the column s is 1.

Theorem 2. *Let $t \in C$ be a point of weight 3. Then the identities*

$$p_{k+1,k}(t) = p_{k+1,k}^{(1,i_1)}(t) + p_{k+1,k}^{(1,i_2)}(t) + p_{k+1,k}^{(1,i_3)}(t)$$

hold true for every $k = 3, 4, \ldots, 8$.

Proof. Let $z \in C$ have weight k and distance $k+1$ from t. Then the equality $d(z,t) = wt(z) + wt(t) - 2wt(z * t)$ implies that $wt(z * t) = 1$, i.e. exactly one of the columns determined by the support of t, intersects z in 1. □

Remark. Of course, analogous assertion is true for other weight of t.

We also use the following assertion which was proved for the 2-block in [2] and analogous proof follows for the 3-block.

We consider the point $\mathbf{t}' \in C'$, obtained from t after cutting one column of C (so C' is the resulting $(4, 8, 96)$ BOA) which passes through an 1 from the support of t. Then we apply Algorithm B for $\mathbf{t}'$ in C' and compare the results as follows. Denote by $R_{i,j}$ the set of the points of C' of weight j and at distance i from $\mathbf{t}'$ and let $|R_{i,j}| = r_{i,j}(t)$. The numbers $r_{i,j}(t)$ will now come from two directions to be compared – from Algorithm B for C' and $\mathbf{t}'$ and the relations between $\mathbf{t}'$ and t.

Theorem 3. *The numbers $r_{i,j}$ from Algorithm B also satisfy the equalities $r_{i,j}(t) = p_{i+1,j}^{(0)}(t) + p_{i,j+1}^{(1)}(t)$ for every $i, j \in \{0, 1, \ldots, 8\}$.*

We use Theorems 1-3 in the following way. In the beginning we consider all $\binom{9}{3} = 84$ points of weight 3. All 14 points at distance 1 from y_1 or y_2 should be removed because there are no points of C at distance 1 to y_1 nor to y_2. Then we partition the remaining 70 points into subclasses according to their supports. For every subclass we apply Theorems 1-3 to decide if that subclass is admissible.

We consider three cases according to the results from [2]. In every case the 0-, 1- and 2-blocks of C are

$$\begin{aligned}
\mathbf{0} &= 000000000 \\
\mathbf{x} &= 100000000 \\
\mathbf{y_1} &= 011000000 \\
\mathbf{y_2} &= 000110000,
\end{aligned}$$

but the distance distributions of the points $\mathbf{0}$, $\mathbf{x}$, $\mathbf{y_1}$ and $\mathbf{y_2}$ are different.

We shall use the following notation for distance distributions

$$\begin{aligned} W_1 &= (1,1,2,23,26,15,18,9,1,0), \\ W_2 &= (1,1,3,19,31,15,13,13,0,0), \\ U_1 &= (1,0,5,22,21,20,19,6,2,0), \\ U_2 &= (1,0,6,18,26,20,14,10,1,0) \end{aligned}$$

(the i-th coordinate gives the number of the points of C which are at distance $i-1$).

In all cases the distance distribution of $\mathbf{0}$ is W_1.

Case 1. The distance distributions of $\mathbf{x}$, $\mathbf{y_1}$ and $\mathbf{y_2}$ are W_1, U_1 and U_1, respectively.

We first consider the points of the 3-block with respect to their first coordinate – 0 or 1. Since $x_3^{(1)} = 2$, there are only 2 out of 28 such points belonging to the 3-block.

Eleven points fail to pass the test: 110100000 (for all possibilities for one 1 in coordinates 2-3, one 1 in coordinate 4-5, 4 such points in total), 100001001 (for all possibilities for one 1 in coordinates 6-7-8, 3 such points in total), 100001100 (all possibilities for two 1's in coordinates 6-7-8, 3 such points in total) and the point 000001110.

The points below pass the test of Theorem 1 and thus 23 of them constitute the 3-block of C. More precisely, the 3-block consists of 2 points from A1-2 and 21 points from B1-5.

A1. The points 110001000, all possibilities for one 1 in coordinates 2-3-4-5, one 1 in coordinates 6-7-8, 12 such points in total.

A2. The points 110000001, all possibilities for one 1 in coordinates 2-3-4-5, 4 such points in total. the points 000001101 (all possibilities for the two 1's in coordinates 6-7-8, 3 such points in total),

B1. The points 010101000, all possibilities for one 1 in coordinates 2-3, one 1 in coordinates 4-5, one 1 in coordinates 6-7-8, 12 such points in total.

B2. The points 010001100, all possibilities for one 1 in coordinates 2-3-4-5, two 1's in coordinates 6-7-8, 12 such points in total.

B3. The points 010100001, all possibilities for one 1 in coordinates 2-3, one in coordinates 4-5, 4 such points in total.

B4. The points 010001001, all possibilities for one 1 in coordinates 2-3-4-5, one 1 in coordinates 6-7-8, 12 such points in total.

B5. The points 000001101, all possibilities for 2 ones in coordinates 6-7-8, 3 such points in total.

Denote by a_i, $i = 1, 2$, and b_j, $j = 1, 2, \ldots, 5$, the number of the points in **Ai.** and **Bi**, respectively. These numbers satisfy the following equalities:

$a_1 + a_2 + b_1 + b_2 + b_3 + b_4 + b_5 = 23$ (the size of the 3-block);

$a_1 + a_2 = 2$ (follows from $x_3^{(1)} = 2$);

$a_1 + a_2 + 2b_1 + b_2 + 2b_3 + b_4 = 4.7 = 28$ (follows from $x_3^{(i)} = 7$ for $i = 2, 3, 4, 5$);

$a_1 + b_1 + 2b_2 + b_4 + 2b_5 = 3.10 = 30$ (follows from $x_3^{(i)} = 10$ for $i = 6, 7, 8$);

$a_2 + b_3 + b_4 + b_5 = 9$ (follows from $x_3^{(9)} = 9$).

Further, we write $a_1 = a_{1,1} + a_{1,2}$, $a_2 = a_{2,1} + a_{2,2}$, $b_2 = b_{2,1} + b_{2,2}$ and $b_4 = b_{4,1} + b_{4,2}$, where $a_{1,1}$ and $a_{1,2}$ correspond to one's in coordinates 2-3 and 4-5, respectively, and analogously for $a_2 = a_{2,1}$, $a_{2,2}$, $b_{2,1}$, $b_{2,2}$, $b_{4,1}$, $b_{4,2}$. Then our knowledge of the structure of the 3-block gives further equations:

$a_{1,1} + a_{2,1} + b_1 + b_{2,1} + b_3 + b_{4,1} = p_{3,3}(y_1) = 14$;

$a_{1,2} + a_{2,2} + b_1 + b_{2,2} + b_3 + b_{4,2} = p_{3,3}(y_2) = 14$;

$a_{1,1} + a_{2,1} = p_{3,3}^{(1,1)}(y_1) = 1$

$a_{2,1} + b_3 + b_{4,1} = p_{3,3}^{(1,9)}(y_1) \in \{4, 5\}$ and $a_{2,2} + b_3 + b_{4,2} = p_{3,3}^{(1,1)}(y_2) \in \{4, 5\}$;

$a_{1,1} + b_1 + 2b_{2,1} + b_{4,1} = p_{3,3}^{(1,6)}(y_1) + p_{3,3}^{(1,7)}(y_1) + p_{3,3}^{(1,8)}(y_1) \in \{12, 13, \ldots, 18\}$;

$a_{1,2} + b_1 + 2b_{2,2} + b_{4,2} = p_{3,3}^{(1,6)}(y_2) + p_{3,3}^{(1,7)}(y_2) + p_{3,3}^{(1,8)}(y_2) \in \{12, 13, \ldots, 18\}$.

$b1 + b3 = p_{3,3}^{(1,4)} + p_{3,3}^{(1,5)} \in \{6, 7, 8, 9, 10\}$.

The above unknowns are nonnegative integer and satisfy the following inequalities: $a_{1,1} \leq a_1 \leq 2$, $a_{1,2} \leq a_1 \leq 2$, $a_{2,1} \leq a_2 \leq 2$, $a_{2,2} \leq a_2 \leq 2$, $b_1 \leq 12$, $b_{2,1} \leq 6$, $b_{2,2} \leq 6$, $b_3 \leq 4$, $b_{4,1} \leq 6$, $b_{4,2} \leq 6$ and $b_5 \leq 3$. Unfortunately these conditions are still not enough for contradiction and we obtained 48 solutions.

We also need to consider two cases coming from the results from [2] by investigations of the 1- and 2-blocks.

Case 2. The distance distributions of $\mathbf{x}$, $\mathbf{y_1}$ and $\mathbf{y_2}$ are W_2, U_1 and U_2, respectively.

Case 3. The distance distributions of $\mathbf{x}$, $\mathbf{y_1}$ and $\mathbf{y_2}$ are W_2, U_1 and U_1, respectively.

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