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New asymptotic bounds for some spherical $(2k-1)$-designs

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Spherical designs

Definition. A spherical τ-design $C \subset \mathbb{S}^{n-1}$ is a finite nonempty subset of $\mathbb{S}^{n-1}$ such that for every point $x \in \mathbb{S}^{n-1}$ and for every real polynomial $f(t)$ of degree at most τ, the equality

$$\sum_{y \in C} f(\langle x, y \rangle) = f_0 |C|. \tag{1}$$

holds, where f_0 is the first coefficient in the expansion $f(t) = \sum_{i=0}^{k} f_i P_i^{(n)}(t)$ in terms of the Gegenbauer polynomials.
We have the following formula

$$f_0 = a_0 + \sum_{i=1}^{[k/2]} \frac{a_{2i}(2i-1)!!}{n(n+2)\cdots(n+2i-2)} = a_0 + \frac{a_2}{n} + \frac{3a_4}{n(n+2)} + \cdots .$$

Delsarte-Goethals-Seidel,
[Spherical codes and designs, *Geom. Dedicata* 6, 1977, 363-388.]

Problems

Problem A. For fixed strength $\tau \geq 3$ and dimension $n \geq 3$ find bounds on the quantity

$$B_{\mathsf{odd}}(n,\tau) = \min\{M = |C| : |C| \text{ is odd and } \exists\ \tau\text{-design } C \subset \mathbb{S}^{n-1}\}.$$

Problem B. Decide whether a τ-design on $\mathbb{S}^{n-1}$ of cardinality $M = |C|$ exists for fixed strength τ, dimension n and M.

Our techniques

Let $C \subset \mathbb{S}^{n-1}$ be a spherical τ-design. For every point $x \in \mathbb{S}^{n-1}$ we consider the inner products of x with all points of C, i.e. the (possibly) multiset

$$I(x) = \{\langle u, x \rangle : u \in C\}.$$

There exists an ordering of the points in $C = \{u_1, u_2, \ldots, u_{|C|}\}$
so that $-1 \leq \langle u_1, x \rangle \leq \langle u_2, x \rangle \leq \cdots \leq \langle u_{|C|}, x \rangle \leq 1$.
Let $t_i(x) = \langle u_i, x \rangle$ for $i = 1, 2, \ldots, |C|$.
Then we consider

$$I(x) = \{t_1(x), t_2(x), \ldots, t_{|C|}(x)\},$$

where $-1 \leq t_1(x) \leq t_2(x) \leq \cdots \leq t_{|C|}(x) \leq 1$,
as $t_{|C|}(x) = 1 \Leftrightarrow x \in C$.

Our techniques

Then equation (1) gives

$$\sum_{i=1}^{|C|-1} f(t_i(x)) = f_0|C| - f(1), \tag{2}$$

for $\forall x \in C$ and $\forall f(t) \in \mathbb{R}[t]$, $\deg(f) \leq \tau$.

Let $C \subset \mathbb{S}^{n-1}$ be a spherical τ-design. For every point $x \in C$ we denote by $U_{\tau,i}(x)$ (respectively $L_{\tau,i}(x)$) for any upper (resp. lower) bound on the inner product $t_i(x)$, i.e.

$$L_{\tau,i}(x) \leq t_i(x) \leq U_{\tau,i}(x).$$

When a bound does not depend on x we omit x in the notation ($L_{\tau,i}$ and $U_{\tau,i}$).

Our techniques

For a fixed (relatively small) strength τ in
[A method for proving nonexistence of spherical designs of odd strength and odd cardinality, *Problems of Information Transmission*, 2009, vol. 45:2, 41-55.] Boumova-Boyvalenkov-Stoyanova proved the nonexistence of spherical designs in many cases for (relatively small) odd cardinality.

We consider $(2k-1)$-designs $C \subset \mathbb{S}^{n-1}$ with odd cardinality $|C|$ such that the conditions

$$2 \leq \rho_0|C| < 3 \quad \text{and} \quad \alpha_{k-1} < 2\alpha_0^2 - 1 \tag{3}$$

are satisfied (ρ_0, α_0, α_{k-1} are defined by Levenshtein).

Our techniques

There exists a special triple $\{x, y, z\} \subset C$ such that

$$\langle x, y \rangle = t_1(x) = t_1(y) \leq \alpha_0$$

and $$\langle x, z \rangle = t_2(x) = t_1(z) \leq \alpha_0.$$

Every special triple $\{x, y, z\}$ is extended to a special quadruple $\{x, y, z, u\}$ by adding the point $u \in C$ which is defined by $t_2(z) = \langle z, u \rangle$. Our method is based on the careful investigation of the special quadruples $\{x, y, z, u\} \subset C$.

A special quadruple $\{x, y, z, u\} \subset C$ is called "good " if $t_2(z) \leq \alpha_0$.

Our main purpose is to obtain a bound $t_1(z) \leq U_{\tau,1}(z) < \alpha_0$. Such bounds start a procedure (of improving other bounds) which often reaches a contradiction with the existence of C. The inequality $U_{\tau,1}(z) < \alpha_0$ can be obtained in all cases: when a special quadruple which is not "good" exists, and when all special quadruples are "good".

Our techniques

If all special quadruples are "good" then there are stronger conditions – there exists a special "good" quadruple $\{x, y, z, u\} \subset C$ such that
$t_{|C|-2}(x) \geq 2\alpha_0^2 - 1 \longleftrightarrow$ we call it x-"good"
or
$t_{|C|-2}(z) \geq 2\alpha_0^2 - 1 \longleftrightarrow$ we call it z-"good", respectively.

In both cases (existence of at least one quadruple which is not "good" and all quadruples are "good"), we obtain a new necessary condition for the existence of the designs under consideration. These conditions are determined as two existence checks $L_x(g)$ and $L_z(g)$.

The asymptotic case

Problem. For a fixed odd integer $\tau = 2k-1 \geq 3$ and for $n \to \infty$ obtain lower bounds of $B_{\text{odd}}(n,\tau)$,

$$B_{\text{odd}}(n,\tau) = \min\{|C| : |C| \text{ is odd and } \exists\ \tau\text{-designs } C \subset \mathbb{S}^{n-1}\}.$$

More precisely, we want to find bounds

$B_{\text{odd}}(n,\tau) \gtrsim An^{k-1}$, $(A = const.)$,

which means $\underline{\lim}_{n\to\infty} \frac{B_{\text{odd}}(n,\tau)}{n^{k-1}} \geq A$.

Some known results

The spherical $(2k-1)$-designs – the asymptotic case

We consider spherical $(2k-1)$-designs $C \subset \mathbb{S}^{n-1}$ with odd cardinality $|C| = (\frac{2}{(k-1)!} + \gamma)n^{k-1}$ as $n \to \infty$, where γ is a positive constant.

In [Spherical codes and designs, *Geom. Dedicata* 6, 1977, 363-388.] Delsarte-Goethals-Seidel obtained

$$B_{\mathsf{odd}}(n, 2k-1) \gtrsim \frac{2}{(k-1)!} n^{k-1}.$$

In [Nonexistence of certain spherical designs of odd strengths and cardinalities, *Discr. Comp. Geom.* 21, 1999, 143-156], Boyvalenkov-Danev-Nikova obtained

$$B_{\text{odd}}(n, 2k-1) \gtrsim \frac{(1+\sqrt[2k-1]{2})}{(k-1)!} n^{k-1}.$$

This asymptotic corresponds also to the necessary condition

$$\rho_0 |C| \geq 2.$$

for the existence of the designs under consideration, which is obtained in [Necessary conditions for existence of some designs in polynomial metric spaces, *Europ. J. Combin.* 20, 1999, 213-225], by Boumova-Boyvalenkov-Danev.

In [New nonexistence results for spherical designs, in *Constructive Theory of Functions* (B. Bojanov, Ed.) Darba, Sofia 2003, 225-232], Boumova-Boyvalenkov-Danev obtained

$$B_{\mathsf{odd}}(n, 2k-1) \gtrsim \frac{(1+x_0)}{(k-1)!} n^{k-1},$$

where $x_0 > \sqrt[(2k-1)]{2}$ is the root of the equation

$$2(x^{4k-2} + (2-x^2)^{2k-1})^{2k-2} = x^{4k-3}(x^{4k-3} - (2-x^2)^{2k-2})^{2k-2}.$$

Therefore, we can assume that $\gamma \geq \frac{x_0-1}{(k-1)!}$.

The parameters' behavior in our asymptotic process is as follows

$$\alpha_0 \approx -\frac{1}{1+\gamma(k-1)!}, \quad \alpha_1 \approx 0, \ldots, \alpha_{k-1} \approx 0 \text{ and}$$

$$\rho_0|C| \approx (1+\gamma(k-1)!)^{2k-1}.$$

The conditions (3) give

$$\rho_0|C| < 3 \iff \gamma < \frac{\sqrt[2k-1]{3}-1}{(k-1)!},$$

$$\alpha_{k-1} < 2\alpha_0^2 - 1 \iff \gamma < \frac{\sqrt{2}-1}{(k-1)!}.$$

Therefore, we assume that $\gamma \in [\frac{x_0-1}{(k-1)!}, \frac{\sqrt[2k-1]{3}-1}{(k-1)!}]$.

Let $\tau = 2k-1 \geq 3$ be fixed, $n \to \infty$ and
$|C| = (\frac{2}{(k-1)!} + \gamma)n^{k-1}$, where $\gamma \in [\frac{x_0-1}{(k-1)!}, \frac{\sqrt[2k-1]{3}-1}{(k-1)!}]$.

Let $\{x, y, z, u\} \subset C$ be a special quadruple.

General bounds on the inner products of $I(x)$ and $I(z)$:

$$L_{2k-1,1}(z) \leq t_1(z) \leq U_{2k-1,1}(z) = \alpha_0,$$
$$t_1(z) \leq t_2(z) \leq U_{2k-1,2}(z),$$
$$L_{2k-1,3}(z) \leq t_3(z),$$
$$t_{|C|-1}(z) \geq L_{2k-1,|C|-1}(z) = 2\alpha_0^2 - 1,$$
$$t_{|C|-1}(x) \geq L_{2k-1,|C|-1}(x).$$

Case 1. There exists at least one special quadruple which is not "good", i.e. $t_2(z) > \alpha_0$ for this special quadruple.

$$t_1(z) \leq U_{2k-1,1}(z) < \alpha_0,$$
$$t_1(z) = t_2(x) \leq U_{2k-1,2}(x) = U_{2k-1,1}(z),$$

$$t_{|C|-1}(z) \geq L_{2k-1,|C|-1}(z) = 2U_{2k-1,1}(z)^2 - 1,$$

$$t_{|C|-1}(x) \geq L_{2k-1,|C|-1}(x),$$

$L_x^{(1)}(g) < 0 \Rightarrow \nexists C$ in Case 1.

Case 2. All special quadruples are "good", i.e. $t_2(z) \leq \alpha_0$ in all special quadruples.
Case 2.1. There exists an x-"good" special quadruple.

We use $t_2(z) \leq \alpha_0$ to improve the bound $L_{2k-1,3}(z) \leq t_3(z)$,

$$t_1(z) \leq U_{2k-1,1}(z) < \alpha_0,$$
$$t_1(z) = t_2(x) \leq U_{2k-1,2}(x) = U_{2k-1,1}(z),$$

$$t_{|C|-1}(z) \geq L_{2k-1,|C|-1}(z) = 2U_{2k-1,1}(z)^2 - 1,$$

$$t_{|C|-2}(x) \geq 2\alpha_0^2 - 1,$$

$$t_{|C|-1}(x) \geq L_{2k-1,|C|-1}(x),$$

$L_x^{(2)}(g) < 0 \Rightarrow \nexists C$ in Case 2.1.

Case 2. All special quadruples are "good", i.e. $t_2(z) \leq \alpha_0$ in all special quadruples.
Case 2.2. There exists a z-"good" special quadruple.

Using $t_2(z) \leq \alpha_0$ we obtain a better bound $L_{2k-1,3}(z) \leq t_3(z)$,

$$t_1(z) \leq U_{2k-1,1}(z) < \alpha_0,$$
$$t_1(z) = t_2(x) \leq U_{2k-1,2}(x) = U_{2k-1,1}(z),$$

$$t_{|C|-2}(z) \geq 2\alpha_0^2 - 1,$$

$$t_{|C|-1}(z) \geq L_{2k-1,|C|-1}(z) = 2U_{2k-1,1}(z)^2 - 1,$$

$$t_{|C|-1}(x) \geq L_{2k-1,|C|-1}(x),$$

$L_x^{(3)}(g) < 0 \Rightarrow \nexists C$ in Case 2.2.

It follows from the corresponding Lemmas and their proofs that all our bounds are monotonic in the right direction – the lower bounds are increasing and the upper bounds are decreasing.

Also, the functionals $L_x(g)$ and $L_z(g)$ are decreasing (note that $g(t)$ is the optimal polynomial for our asymptotic).

Therefore the nonexistence proof for any admissible γ_0 means nonexistence for every admissible $\gamma < \gamma_0$.

The best γ_0 we have achieved is $\gamma_0 = \dfrac{\sqrt[2k-1]{3} - 1}{(k-1)!}$.

We obtain the following new asymptotic bound for the minimum possible odd cardinality of spherical $(2k-1)$-designs.

Theorem. If $C \subset \mathbb{S}^{n-1}$ is a spherical τ-design, $\tau = 2k-1$, $k = 3, 4, \ldots, 13$, of odd cardinality and n is large enough, then $\rho_0 |C| \geq 3$. In other words,

$$B_{\mathrm{odd}}(n, 2k-1) \gtrsim \frac{(1 + \sqrt[2k-1]{3})}{(k-1)!} n^{k-1}.$$

New results

The spherical $(2k-1)$–designs – the asymptotical case

τ	Delsarte-Goethals-Seidel bounds	Previously best known bounds	New bounds (with $\gamma = \frac{2k-1\sqrt{3}-1}{(k-1)!}$)
3	$2n$	$2.3925n$ $[BBKS]$	
5	n^2	$1.09309n^2$ $[BBD]$	$1.12286n^2$
7	$\frac{n^3}{3} \approx 0.33333n^3$	$0.35314n^3$ $[BBD]$	$0.36165n^3$
9	$\frac{n^4}{12} \approx 0.08333n^4$	$0.08667n^4$ $[BDN, BBD]$	$0.08874n^4$
11	$\frac{n^5}{60} \approx 0.01666n^5$	$0.01721n^5$ $[BDN, BBD]$	$0.01754n^5$
13	$\frac{n^6}{360} \approx 0.0027777n^6$	$0.0028538n^6$ $[BDN, BBD]$	$0.0029003n^6$
15	$\frac{n^7}{2520} \approx 0.0003968n^7$	$0.0004062n^7$ $[BDN, BBD]$	$0.0004119n^7$
17	$\frac{n^8}{20160} \approx 0.00004960n^8$	$0.00005063n^8$ $[BDN, BBD]$	$0.00005126n^8$

Table 1. Asymptotic lower bounds for $B_{\mathsf{odd}}(n, \tau)$.

THANK YOU FOR YOUR ATTENTION !