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On the classification of binary self-dual \([44, 22, 8]\) codes with an automorphism of order 3

St. Bouyuklieva
*Veliko Tarnovo University, Bulgaria*

R. Russeva, N Yankov, N. Ziapkov and M. Nikolova
*Shumen University, Bulgaria*
\( C - [n, k, d] \) binary linear code

\( C^\perp \) is the dual code of \( C \) under the standard inner product

\( C \) is a self-dual code if \( C = C^\perp \)

For SD codes \( n = 2k \) and all weights are even

Automorphism of the code \( C \) is a permutation of the coordinates that preserves \( C \)

\( \text{Aut}(C) \) - the group of all automorphisms of \( C \)

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\( \mathcal{C} - [44, 22, 8] \) SD binary code

\[ p = 11, 7, 5, 3 \] - odd primes dividing \( |Aut(\mathcal{C})| \)

The codes having automorphisms of prime orders \( p \geq 5 \) are classified

Yorgov(1993), Yorgov and Russeva(1994)

We investigate codes with an automorphism of order 3
\[ \mathcal{C} - \text{[44, 22, 8] SD code with an automorphism } \sigma \text{ of type 3-}(c,f) \]

\[ \sigma - \text{with } c \text{ cycles of order 3 and } f = 44-3c \]

\[ c = 6, 8, 10, 12 \text{ and } 14 \]

\[ c = 6 \text{ and } 14 \text{ (Bouyuklieva (2004), Yankov (2007))} \]

We classify all codes with an automorphism of order 3 with 8, 10 and 12 3-cycles

We complete the classification of SD [44,22,8] codes with an automorphism of odd prime order
Construction Method

\[ F_\sigma(C) = \{ v \in C : v\sigma = v \} \] - the fixed part of \( C \)

\[ E_\sigma(C) \] - the set of vectors in \( C \) with even weight in each 3-cycle of \( \sigma \) and 0 at the fixed points

\[ C = F_\sigma(C) \oplus E_\sigma(C) \]

\[ \text{gen}(C) = \left( \begin{array}{c} \text{gen}(F_\sigma(C)) \\ \text{gen}(E_\sigma(C)) \end{array} \right) \]
\[ \pi : F_\sigma(C) \rightarrow F_2^{c+f} \]

\( \pi(v) \) - vector of length \( c + f \) obtained by choosing a coordinate from each cycle and fixed points of \( v \).

\( \pi(F_\sigma(C)) \) - a binary self-dual \([c + f, \frac{c+f}{2}]\) code
$E_\sigma(C)^*$ - the code $E_\sigma(C)$ with the last $f$ coordinates deleted.

$P = \langle x + 1 \rangle$ - cyclic code of length 3

$P \cong F_4$ - set of even weight polynomials

The restriction of $v \in E_\sigma(C)$ on any 3-cycle of $\sigma$ can be viewed as an element from $P$

$(v_0, v_1, v_2) \rightarrow v_0 + v_1x + v_2x^2 \in P$

$\varphi : E_\sigma(C)^* \rightarrow P^c$. 
Theorem 1 A code $C$ with an automorphism $\sigma$ of order 3 is SD if and only if:

i) $\pi(F_\sigma(C))$ is a $[c + f, \frac{c+f}{2}]$ binary SD code;

ii) $\varphi(E_\sigma(C)^*)$ is a quaternary Hermitian SD code of length $c$
Two weight enumerators for [44, 22, 8] SD codes:

\[ W_{44,1}(y) = 1 + (44 + 4\beta)y^8 + (976 - 8\beta)y^{10} + \ldots \]

for 10 ≤ β ≤ 122

\[ W_{44,2}(y) = 1 + (44 + 4\beta)y^8 + (1232 - 8\beta)y^{10} + \ldots \]

for 10 ≤ β ≤ 154

\( W_{44,1} \) and \( W_{44,2} \) for various \( \beta \)

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Codes with automorphism of type $3 - (8, 20)$

$\varphi(E_\sigma(C)^*) = C_\varphi$ - unique quaternary $[8, 4, 4]$ code

$\pi(F_\sigma(C)) = C_\pi$ is a $[28, 14, \geq 4]$ SD code
\[ G_\pi = \begin{pmatrix} B & O \\ O & D \\ E & F \end{pmatrix} \]

\( B \) generates \([8, k_1, \geq 4]\) SO code
\( D \) generates \([20, 6 \leq k_1 + 6 \leq 10, \geq 8]\) SO code

All optimal binary self-orthogonal codes of length 20 are classified \((\text{Bouyuklieva}(2004))\)

23 - \([20,6,8]\) SO codes; 4 - \([20,7,8]\) SO codes; unique \([20,8,8]\) SO code

Hence \(k_1 \leq 2\)
Theorem 2 There are exactly 4570 inequivalent [44, 22, 8] SD codes with automorphism of type 3-(8,20).

Their weight enumerators are of both types $W_{44,1}$ and $W_{44,2}$ with $\beta \leq 76$
Codes with automorphism of type $3 - (10, 14)$

$C_\varphi$ - Hermitian $[10,5,4]$ code: $E_{10}$ or $B_{10}$

$C_\pi$ is a $[24, 12, \geq 4]$ SD binary code

Seven such codes can be used, namely: $G_{24}$, $R_{24}$, $U_{24}$, $W_{24}$, $X_{24}$, $Y_{24}$ and $Z_{24}$

We fix the generator matrix of the subcode $C_\varphi$ and consider all possibilities for the generator matrix of the subcode $C_\pi$
Theorem 3 There are exactly 8738 inequivalent $[44, 22, 8]$ SD codes with automorphism of type 3-(10,14).

1815 codes with $W_{44,1}$ for $\beta \leq 62$
6923 codes with $W_{44,2}$ for $\beta \leq 24$
Codes with automorphism of type $3 - (12, 8)$

$C_\varphi$ - Hermitian $[12,6,4]$ code: $d_{12}, 2d_6, 3d_4, e_6 \oplus e_6$, and $e_7 + e_5$

$C_\pi$ is a $[20, 10, \geq 4]$ SD binary code

Seven such codes, namely: $d_{12} + d_8$, $d_{12} + e_8$, $d_{20}$, $d_5^5$, $d_6^3 + f_2$, $d_8^2 + d_4$, and $e_7^2 + d_6$
Theorem 4 There are exactly $123147$ inequivalent $[44, 22, 8]$ SD codes with automorphism of type $3-(12,8)$.

Their weight enumerators are of both types

Codes with $W_{44,1}$ for $\beta = 10, \ldots, 68, 70, 72, 74, 82, 86, 90, 122$

Codes with $W_{44,2}$ for $\beta = 0, \ldots, 56, 58, \ldots, 62, 64, 66, 68, 70, 72, 74, 76, 82, 86, 90, 104, 154$
For computing codes - **GAP 4.4** and

**Q-Extensions (I. Bouyukliev)**
THANK YOU!