Minimal lengths for codes with given primal and dual distance

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1 Motivation and background
2 Definitions and notations
3 Objectives
4 History of the problem
5 Preliminaries
6 Computer tools & techniques
7 Two examples
8 Table of results
1 Motivation and background
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3 Objectives
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5 Preliminaries
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7 Two examples
8 Table of results
Contents

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5 Preliminaries
6 Computer tools & techniques
7 Two examples
8 Table of results
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2 Definitions and notations
3 Objectives
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5 Preliminaries
6 Computer tools & techniques
7 Two examples
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5 Preliminaries
6 Computer tools & techniques
7 Two examples
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3 Objectives
4 History of the problem
5 Preliminaries
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7 Two examples
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Motivation and background

In cryptography, in order to obscure the relationship between the ciphertext and the key, substitution boxes (S-boxes) are generally used to transform $S$ input bits into $T$ output bits.

An S-box is a collection of $T$ Boolean functions $f : GF(2)^S \rightarrow GF(2)$.

The security of a block cipher against various attacks comes down to the security of the S-Boxes, which in turn comes down to the security of the Boolean functions.
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The security of a block cipher against various attacks comes down to the security of the S-Boxes, which in turn comes down to the security of the Boolean functions.
Definition

A Boolean function $f : \mathbb{GF}(2)^S \rightarrow \mathbb{GF}(2)$ is called **K-resilient** if we can fix any set of $K$, $K < S$, input bits and the function gives 0 and 1 equally often, on the remaining $2^{S-K}$ different inputs.

Definition

A Boolean function $f : \mathbb{GF}(2)^S \rightarrow \mathbb{GF}(2)$ is said to satisfy propagation criteria, $PC(L)$ if for a fixed $x \in \mathbb{GF}(2)^S$

$$f(x) - f(x + \Delta)$$

gives 0 and 1 equally often, for $\Delta \in \mathbb{GF}(2)^S$ with Hamming weight $1 \leq w(\Delta) \leq L$
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A Boolean function $f : GF(2)^S \rightarrow GF(2)$ is said to satisfy the extended propagation criteria, $EPC(L)$ of order $K$ if

$$f(x) - f(x + \Delta)$$

is $K$-resilient for $\Delta \in GF(2)^S$ with $1 \leq w(\Delta) \leq L$.

In fact, it has been shown that the $EPC(L)$ of order $K$ is directly related to security of a Boolean function against both linear and differential attacks.
Motivation and background

Question:
Given $L$ and $K$, what is the minimum $S$ for which an $EPC(L)$ of order $K$ function exists?

Theorem (Kurosawa and Satoh(1997))

There exists an $EPC(L)$ function $f(x_1, ..., x_S)$ of order $K$ if there exists a linear code of length $\frac{S}{2}$, some dimension, minimum distance $K + 1$ and dual distance $L + 1$.

If we let $n = \frac{S}{2}$, $d = K + 1$, $d^\perp = L + 1$ and let $k$ denote the dimension we can reformulate the question.
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Definitions and Notations

Reformulated question:
What is the least $n$ such that there exists a linear code of length $n$ with minimum distance $d$ and dual distance $d^\perp$, where $d$ and $d^\perp$ are fixed?

Definition (Matsumoto et.al. 2004)

$N(d, d^\perp) = \text{The minimum } n \text{ such that there exists a linear } [n, k, d] \text{ code with dual distance } d^\perp.$
Reformulated question:
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Objectives

- Find some values for $N(d, d^\perp)$ for specific $d$ and $d^\perp$.
- For these values classify all inequivalent codes reaching $N(d, d^\perp)$. 
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The problem to study the function $N(d, d^\perp)$ was given by Matsumoto et al. in 2006. They presented:

- Some general bounds on the function $N(d, d^\perp)$ (i.e. new versions of known bounds Griesmer, Hamming, linear programming bound).
- Some examples (although no systematical investigation of the exact values of $N(d, d^\perp)$).

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Theorem

Let $C$ be a linear code with minimum distance $d$ and dual distance $d^\perp$, and let $C'$ be the punctured code of $C$. Then $C'$ has minimum distance at least $d - 1$ and dual distance at least $d^\perp$.

For $d, d^\perp > 2$, we have

$$N(d - 1, d^\perp) \leq N(d, d^\perp) - 1$$

$$N(d, d^\perp - 1) \leq N(d, d^\perp) - 1.$$ 

I.e. the $N(d, d^\perp)$ function is strictly increasing in both its arguments.
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I.e. the $N(d, d^\perp)$ function is strictly increasing in both its arguments.
Definition

Let $G$ be a generator matrix of a linear binary $[n, k, d]$ code $C$ and $c \in C$. Then the residual code $\text{Res}(C, c)$ of $C$ with respect to $c$ is the code generated by the restriction of $G$ to the columns where $c$ has a zero entry.

Theorem

Suppose $C$ is a binary $[n, k, d]$ code and suppose $c \in C$ has weight $\omega$, where $d > \omega/2$. Then $\text{Res}(C, c)$ is an $[n - \omega, k - 1, d']$ code with $d' \geq d - \omega + \lceil \omega/2 \rceil$. 
Preliminaries

**Definition**

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Theorem

Suppose C is a binary $[n, k, d]$ code with dual distance $d^\perp$, $c \in C$, and the dimension of Res$(C, c)$ is $k - 1$. Then the dual distance of Res$(C, c)$ is also $d^\perp$. 
We use the program Q_EXTENSION to construct all inequivalent \([n, k, d]\) codes from their residual or shortening codes.

First approach:

Moving backwards through the residuals of a supposed \([n, k, d]^{d\perp}\) code (where the superscript means that the code has dual distance \(d^{\perp}\)) we can extend as:

\[
[k_0, k_0, 1] \rightarrow [n_0, k_0, d_0]^{d\perp} \rightarrow \ldots \rightarrow \\
\rightarrow [n - d, k - 1, \geq d/2]^{d\perp} \rightarrow [n, k, d]^{d\perp}
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(In fact, this does in most cases become a tree of extensions).
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Second approach:

We construct all $[n, k, d]$ codes by extending from their shortened codes. I.e. from codes of the form $[n - i, k - i, d]$ or $[n - i - 1, k - i, d]$. If $G$ is a generator matrix for an $[n - i, k - i, d]$ or an $[n - i - 1, k - i, d]$ code we extend it in all possible ways to

\[
\begin{pmatrix}
\ast & 1_l & i \\
G & 0
\end{pmatrix}
\text{ or }
\begin{pmatrix}
\ast & 1 \\
G & 0
\end{pmatrix}.
\]
Finding $N(9, 5)$ and $N(10, 5)$

From Brouwer’s table we know that there may exist binary $[27, 10, 9]$ and $[28, 10, 10]$ codes with dual distance 5.

If we let $C_{27}$ be a $[27, 10, 9]$ linear code with dual distance 5 we can consider a generator matrix of $C_{27}$ in the form:

$$G_{27} = \begin{pmatrix}
00000 \\
\ldots & G_{22} \\
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Adding a parity check bit to \( G_{27} \) we obtain a generator matrix of a code \( C_{28} \) with parameters \([28, 10, 10]\). This generator matrix has the form:

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G_{28} = \begin{pmatrix}
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(where \( G_{23} \) generates a \([23, 6, 10]\) code).

By exhaustive search we find all inequivalent \([28, 10, 10]\) codes.

The extensions are:

\[ [6, 6, 1] \rightarrow [23, 6, 10](29) \rightarrow [25, 7, 10](30522) \rightarrow [26, 8, 10](507533) \rightarrow [27, 9, 10](30418) \rightarrow [28, 10, 10](10). \]
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Out of these ten, five turn out to have dual distance 5.

\[ N(10, 5) = 28 \] with 5 inequivalent codes.

By deleting each coordinate and analysing the results, we find that there are exactly 137 inequivalent \([27, 10, 9]\) codes with dual distance 5.

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Extensions:

\[ [5, 5, 1] \rightarrow [15, 5, \geq 6](91) \rightarrow [27, 6, 12](178) \rightarrow [28, 7, 12](129) \rightarrow [29, 8, 12](73) \rightarrow [30, 9, 12](9) \rightarrow [31, 10, 12](2) \rightarrow [32, 11, 12](2). \]

The [32, 11, 12] codes turn out to have dual distance 6, which is optimal in the sense that no shorter code, or with different dimension, could achieve this.

Moreover, the [31, 10, 12] codes turn out to have dual distance 5, which is also optimal.

\( N(12, 5) = 31 \) and \( N(12, 6) = 32 \).

Puncturing (and optimality) give us \( N(11, 6) = 31 \), \( N(10, 6) = 30 \) and \( N(9, 6) = 29 \). And also \( N(11, 5) = 30 \).
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Table of the $N(d, d\perp)$ function

<table>
<thead>
<tr>
<th>$d/d\perp$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
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<th>12</th>
</tr>
</thead>
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<tr>
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Thank you for your attention!