# OPTIMAL ARCS IN HJELMSLEV SPACES OF HIGHER DIMENSION

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# 1. Finite chain rings

**Definition.** A ring (associative, $1 \neq 0$, ring homomorphisms preserving 1) is called a **left (right) chain ring** if the lattice of its left (right) ideals forms a chain.

**Theorem.** For a finite ring $R$ with radical $N \neq 0$ the following conditions are equivalent.

(i) $R$ is a left chain ring;

(ii) the principal ideals form a chain;

(iii) $R$ is a local ring and $N = R\theta$ for any $\theta \in N \setminus N^2$;

(iv) $R$ is a right chain ring.

Moreover, if $R$ satisfies the above conditions, every proper left (right) ideal of $R$ has the form $N^i = R\theta^i = \theta^i R$, for some $i \in \mathbb{N}$.

**W.E. Clark, D.A. Drake**, Abh. aus dem Math. Sem. der Univ. Hamburg **39**(1974), 147–153.

**B. McDonald**, *Finite rings with identity*, 1974.

**A. Nechaev**, Mat. Sbornik **20**(1973), 364–382.

**Example. Chain Rings with $q^2$ Elements**

$$\boxed{R\colon |R| = q^2,\ R/\operatorname{rad} R \cong \mathbb{F}_q}$$

$$\boxed{R > \operatorname{rad} R > (0)}$$

R. Raghavendran, Compositio Mathematica **21** (1969), 195–229.

A. Cronheim, Geom. Dedicata **7**(1978), 287–302.

If $q = p^r$ there exist $r + 1$ isomorphism classes of such rings:

- **$\sigma$-dual numbers** over $\mathbb{F}_q$, $\forall \sigma \in \operatorname{Aut} \mathbb{F}_q$: $R_\sigma = \mathbb{F}_q \oplus \mathbb{F}_q t$; addition – componentwise, multiplication –

  $(x_0 + x_1 t)(y_0 + y_1 t) = x_0 y_0 + (x_0 y_1 + x_1 \sigma(y_0))t$;

  $R_\sigma = \mathbb{F}_q[t; \sigma]/(t^2)$.

- the **Galois ring** $\mathrm{GR}(q^2, p^2) = \mathbb{Z}_{p^2}[X]/(f(X))$, $f(X)$ is monic of degree $r$, basic irreducible (irreducible mod $p$).

## 2. Modules over finite chain rings

**Theorem.** Let $R$ be a finite chain ring of nilpotency index $m$. For any finite module ${}_RM$ there exists a uniquely determined partition $\lambda = (\lambda_1 \ldots, \lambda_k) \vdash \log_q|M|$ into parts $\lambda_i \leq m$ such that

$$ {}_RM \cong R/N^{\lambda_1} \oplus \ldots \oplus R/N^{\lambda_k}. $$

The partition $\lambda$ is called the **shape** of ${}_RM$.

The number $k$ is called the **rank** of ${}_RM$.

## 3. Projective Hjelmslev geometries

- $M = R_R^k$;
- $M^* := M \setminus \theta M$;
- $\mathcal{P} = \{xR \mid x \in M^*\}$;
- $\mathcal{L} = \{xR + yR \mid x, y \text{ linearly independent}\}$;
- $I \subseteq \mathcal{P} \times \mathcal{L}$ – incidence relation;
- $\frown$ - **neighbour relation**:

  (N1) $X \frown Y$ if $\exists s, t \in \mathcal{L}$: $X, Y I s, X, Y I t$;

  (N2) $s \frown t$ if $\exists X, Y \in \mathcal{P}$: $X, Y I s, X, Y I t$.

**Definition.** The incidence structure $\Pi = (\mathcal{P}, \mathcal{L}, I)$ with neighbour relation $\bigcirc$ is called the (**right**) **projective Hjelmslev geometry** over the chain ring $R$.

Notation: $\mathrm{PHG}(R_R^k)$

# 4. Multisets of points

Let $\Pi$ be $\mathrm{PG}(k-1,q)$ or $\mathrm{PHG}(R_R^k)$.

**Definition.** A **multiset** in $\Pi = (\mathcal{P}, \mathcal{L}, I)$ is defined as a mapping

$$\mathfrak{K} : \mathcal{P} \to \mathbb{N}_0.$$

• $\mathcal{Q} \subset \mathcal{P}$: $\mathfrak{K}(\mathcal{Q}) = \sum_{x \in \mathcal{Q}} \mathfrak{K}(x)$.

**Definition.** $(n,w)$-**multiarc** in $\Pi$: a multiset $\mathfrak{K}$ with

1) $\mathfrak{K}(\mathcal{P}) = n$;

2) for every hyperplane $H$: $\mathfrak{K}(H) \leq w$;

3) there exists a hyperplane $H_0$: $\mathfrak{K}(H_0) = w$;

# 5. The problem

**Problem.** Given $\nu$, $k$ and a chain ring $R$, find the largest size of a $(\kappa, \nu)$-arc in $\mathrm{PHG}(R_R^k)$.

$m_n(R_R^k)$ – cardinality of the largest $(\kappa, \nu)$-arc in $\mathrm{PHG}(R_R^k)$;

In this paper:

**Problem.** Find $m_\nu(R_R^4)$ for small chain rings $R$ with $|R| = q^2$, $R/\operatorname{rad} R \cong \mathbb{F}_q$.

# 6. A general upper bound

**Theorem.** Let $\mathfrak{K}$ be a $(k,n)$-arc in $\mathrm{PHG}(R_R^3)$ where $|R| = q^2$, $R/N \cong \mathbb{F}_q$. Suppose there exist a point $x$ with $\mathfrak{K}(x) = a$ and a neighbour class of points $[x]$ with $\mathfrak{K}([x]) = b$. Then

$$k \leq (n-a)q^2 + (n-b)q + b.$$

**Theorem.** Let $\mathfrak{K}$ be a $(k,n)$-(multi)arc in $\mathrm{PHG}(R_R^t)$, where $R$ is a chain ring with $|R| = q^2$, $R/\operatorname{rad} R \cong \mathbb{F}_q$, and let $x$ be a point with $\mathfrak{K}(x) = a$. Then

$$k \leq a + m_{n-a}(R_R^{t-1}).$$

**Theorem.** Let $R$ be a chain ring with $|R| = q^2$, $R/\operatorname{rad} R \cong \mathbb{F}_q$. Then

(a) $m_{sq(q+1)}(R_R^3) = sq^2(q^2 + q + 1)$;

(b) $m_{sq(q+1)+1}(R_R^3) = sq^2(q^2 + q + 1) + 1$.

**Theorem.** Let $R$ be a chain ring with $|R| = 4$, $R/\operatorname{rad} R \cong \mathbb{F}_2$. Then

(i) $m_{6t}(R_R^3) = 28t$,

(ii) $m_{6t+1}(R_R^3) = 28t + 1$,

(iii) $m_{6t+2}(R_R^3) = \begin{cases} 28t + 7 & \text{if } R = \mathbb{Z}_4; \\ 28t + 6 & \text{if } R = \mathbb{F}_2[X]/(X^2) \end{cases}$,

(iv) $m_{6t+3}(R_R^3) = 28t + 10$,

(v) $m_{6t+4}(R_R^3) = 28t + 16$,

(vi) $m_{6t+5}(R_R^3) = 28t + 22$,

where $t = 0, 1, 2, \ldots$.

| $n/R$ | $\mathbb{Z}_4$ | $\mathbb{F}_2[X]/(X^2)$ | $n/R$ | $\mathbb{Z}_4$ | $\mathbb{F}_2[X]/(X^2)$ |
|---|---|---|---|---|---|
| 3 | 8 | 6 | 16 | 64 – 67 | 64 – 67 |
| 4 | 10 | 11 | 17 | 65 – 70 | 65 – 70 |
| 5 | 16 | 16 | 18 | 66 – 76 | 66 – 76 |
| 6 | 18 – 23 | 18 – 23 | 19 | 67 – 80 | 75 – 80 |
| 7 | 21 – 29 | 19 – 29 | 20 | 72 – 83 | 76 – 83 |
| 8 | 22 – 30 | 22 – 30 | 21 | 78 – 90 | 78 – 90 |
| 9 | 24 – 36 | 24 – 35 | 22 | 84 – 91 | 90 – 91 |
| 10 | 26 – 39 | 26 – 39 | 23 | 90 – 98 | 90 – 98 |
| 11 | 30 – 45 | 30 – 45 | 24 | 96 – 104 | 96 – 104 |
| 12 | 36 – 51 | 36 – 51 | 25 | 102 – 105 | 105 |
| 13 | 40 – 57 | 40 – 57 | 26 | 108 – 110 | 108 – 110 |
| 14 | 44 – 58 | 44 – 58 | 27 | 114 | 114 |
| 15 | 50 – 61 | 50 – 61 | 28 | 120 | 120 |