On weight distributions of perfect structures

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2. Definitions (mutual distribution of colorings, weight distributions).

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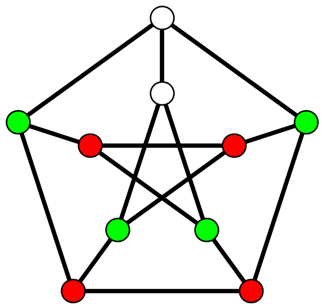
Definition: perfect coloring

Let $G = (V, E)$ be a graph.

Let f be a function ("coloring") on V that possesses exactly k different values $e_0, \ldots, e_{k-1}$ ("colors").

f is called a *perfect coloring* with parameter matrix $S = (S_{ij})_{i,j=0}^{k-1}$, or *$S$-perfect coloring*, iff for any colors e_i, e_j any vertex of color e_i has exactly S_{ij} neighbors of color e_j.

Example: perfect coloring

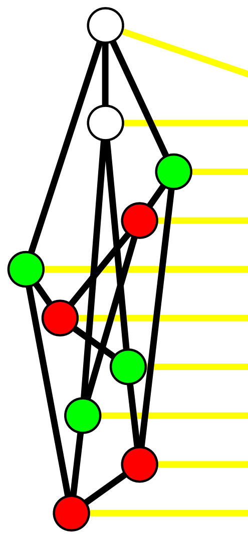


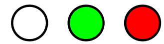
$$S = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix}$$

Equivalent (almost) concepts: perfect coloring; equitable partition; front divisor of the graph; graph covering

Matrix representation of a perfect coloring

f:

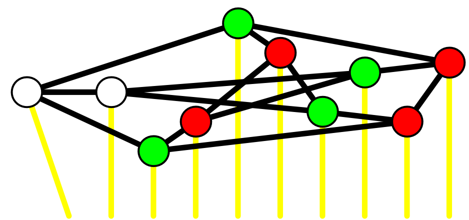


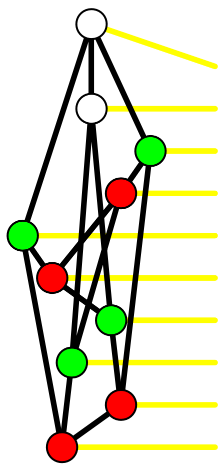


1	0	0
1	0	0
0	1	0
0	0	1
0	1	0
0	0	1
0	1	0
0	1	0
0	0	1
0	0	1

Adjacency matrix

A:





0	**1**	**1**	0	**1**	0	0	0	0	0
1	0	0	0	0	0	**1**	**1**	0	0
1	0	0	**1**	0	0	0	0	**1**	0
0	0	**1**	0	0	**1**	0	**1**	0	0
1	0	0	0	0	**1**	0	0	0	**1**
0	0	0	**1**	**1**	0	**1**	0	0	0
0	**1**	0	0	0	**1**	0	0	**1**	0
0	**1**	0	**1**	0	0	0	0	0	**1**
0	0	**1**	0	0	0	**1**	0	0	**1**
0	0	0	0	**1**	0	0	**1**	**1**	0

Matrix equation for perfect coloring

A – adjacency matrix of the graph;

f – perfect coloring with parameter matrix S.

Then

$$Af = fS$$

Definition: perfect structure

If the equation $Af = fS$ holds for some matrices A, S, and f (of size $N \times N$, $k \times k$, and $N \times k$ respectively) over R, then we will say that f is an *S-perfect structure* (or a *perfect structure* with parameters S) over A.

[1] S. V. Avgustinovich. Perfect structures. Lectures. POSTECH, Korea, May 2007.

Def.: distance coloring, completely regular code

C – some set of vertices (code).

The function $f(x) = e_{d(x,C)}$, where $d(\cdot,\cdot)$ is the natural distance in the graph, is a *distance coloring* with respect to C.

If f is a perfect coloring, then C is called a *completely regular code*.

Examples: completely regular codes

The parameter matrix of a completely regular code is three-diagonal.

– 1-Perfect codes: $\begin{pmatrix} 0 & n \\ 1 & n-1 \end{pmatrix}$; extended: $\begin{pmatrix} 0 & n & 0 \\ 1 & 0 & n-1 \\ 0 & n & 0 \end{pmatrix}$

– Preparata-like codes: $\begin{pmatrix} 0 & n & 0 & 0 \\ 1 & 0 & n-1 & 0 \\ 0 & 2 & n-3 & 1 \\ 0 & 0 & n & 0 \end{pmatrix}$; extended...

– STS: $\begin{pmatrix} 0 & n \\ 3 & n-3 \end{pmatrix}$; SQS: $\begin{pmatrix} 0 & n \\ 4 & n-4 \end{pmatrix}$.

Examples: (non) completely regular codes

– In a *distance-regular graph*, for any vertex x the set $\{x\}$ is a completely regular code.

New. – Every binary $(n = 2^m - 3, 2^{n-m}, 3)$ code (i.e., a code with parameters of doubly shortened Hamming code) is a first color of a perfect coloring with parameters $\begin{pmatrix} 0 & 1 & n-1 & 0 \\ 1 & 0 & n-1 & 0 \\ 1 & 1 & n-4 & 2 \\ 0 & 0 & n-1 & 1 \end{pmatrix}$

Exam.: c. r. codes with large covering radius

– r-dimensional face $\{(x_1, ..., x_r, 0, ..., 0)\}$ in a Hamming graph

– lattice $\{(x_{11}, ..., x_{1r}, x_{21}, \ldots, x_{pr}) \mid \sum_{i=1}^{r} x_{ji} = 0 \forall j\}$ in a Hamming graph of dimension pr.

– p-ary subcube $\{(x_1, ..., x_n) \mid x_i < p\}$ of a q-ary Hamming graph.

Definition: Distribution of one coloring with respect to another

Let f and g be two colorings of the same graph. The matrix $g^T f$ is the *distribution of f with respect to g*. The ijth element is the number of vertices x such that $g(x) = e_i$ and $f(x) = e_j$.

If g is a distance coloring of some C, then $g^T f$ is the *weight distribution of f with respect to C*. The ijth element is the number of vertices x such that $d(x, C) = i$ and $f(x) = e_j$.

Theorem 1 (distribution is a perfect structure)

Let $Af = fS$ and $Ag = gR$ where $A = A^T$. Then

$$R^T(g^T f) = (g^T f)S.$$

I.e., $g^T f$ is an S-perfect structure over R^T.

Proof. $R^T g^T f = (gR)^T f = (A^T g)^T f = g^T Af = g^T fS$. $\square$

Theorem 2 If the matrix $B = \{b_{i,j}\}_{i,j=0}^{n-1}$ is three-diagonal and $b_{i,i+1} \neq 0$, for any $i = 0, ..., n-2$. Then any $\mathbf{S}$-perfect structure $\mathbf{h}$ over B (i.e. $B\mathbf{h} = \mathbf{h}\mathbf{S}$) is uniquely defined by its first row $\mathbf{h}_0$. Moreover, the rows $\mathbf{h}_i$ satisfy the recursive relations

$$\mathbf{h}_i = (\mathbf{h}_{i-1}\mathbf{S} - b_{i-1,i-2}\mathbf{h}_{i-2} - b_{i-1,i-1}\mathbf{h}_{i-1})/b_{i-1,i},$$

and, by induction,

$$\mathbf{h}_i = \mathbf{h}_0 \Pi_i^B(\mathbf{S})$$

where $\Pi_i^B(z)$ is a degree-i polynomial in z.

Corollary

If we have: an $\mathbf{S}$-perfect coloring f, a completely regular code $\mathbf{C}$. Then the weight distribution $\mathbf{h}$ of f with respect to $\mathbf{C}$ is calculated as

$$\mathbf{h}_i = \mathbf{h}_0 \Pi_i^{\mathbf{C}}(\mathbf{S}).$$

Q: How to compute the polynomials $\Pi_i^{\mathbf{C}}$ for C with large covering radius (e.g., $C = \{x\}$)?

Let G be a graph and let for every w from 0 to $diameter(G)$ the matrix $A_w = (a^w_{ij})_{i,j \in V(G)}$ be the distance-w matrix of G (i.e., $a^w_{ij} = 1$ if the graph distance between i and j is w, and $= 0$ otherwise); put $A := A_1$. The graph G is called *distance regular* iff for every w

$$A_w = \Pi_w(A)$$

for some polynomial Π_w of degree w. The polynomials Π_0, Π_1, ..., $\Pi_{diameter(G)}$ are called *P-polynomials* of G.

$Af = fS$, $A^2 f = fS^2$, $A^3 f = fS^3$, ..., and so, $P(A)f = fP(S)$ for any polynomial P. In particular,

$$A_w f = f \Pi_w(S).$$

I.e., the color percentage at the distance w from the vertex i is $f_i \Pi_w(S)$ where f_i = the ith row of f = the color of the vertex i. The weight distribution of f with respect to $\{i\}$ is:

$$(\Pi_0(S) f_i,\ \Pi_1(S) f_i,\ \Pi_2(S) f_i,\ ...)^T;$$

$\Pi_0(S) = Id$, $\Pi_1(S) = S$.

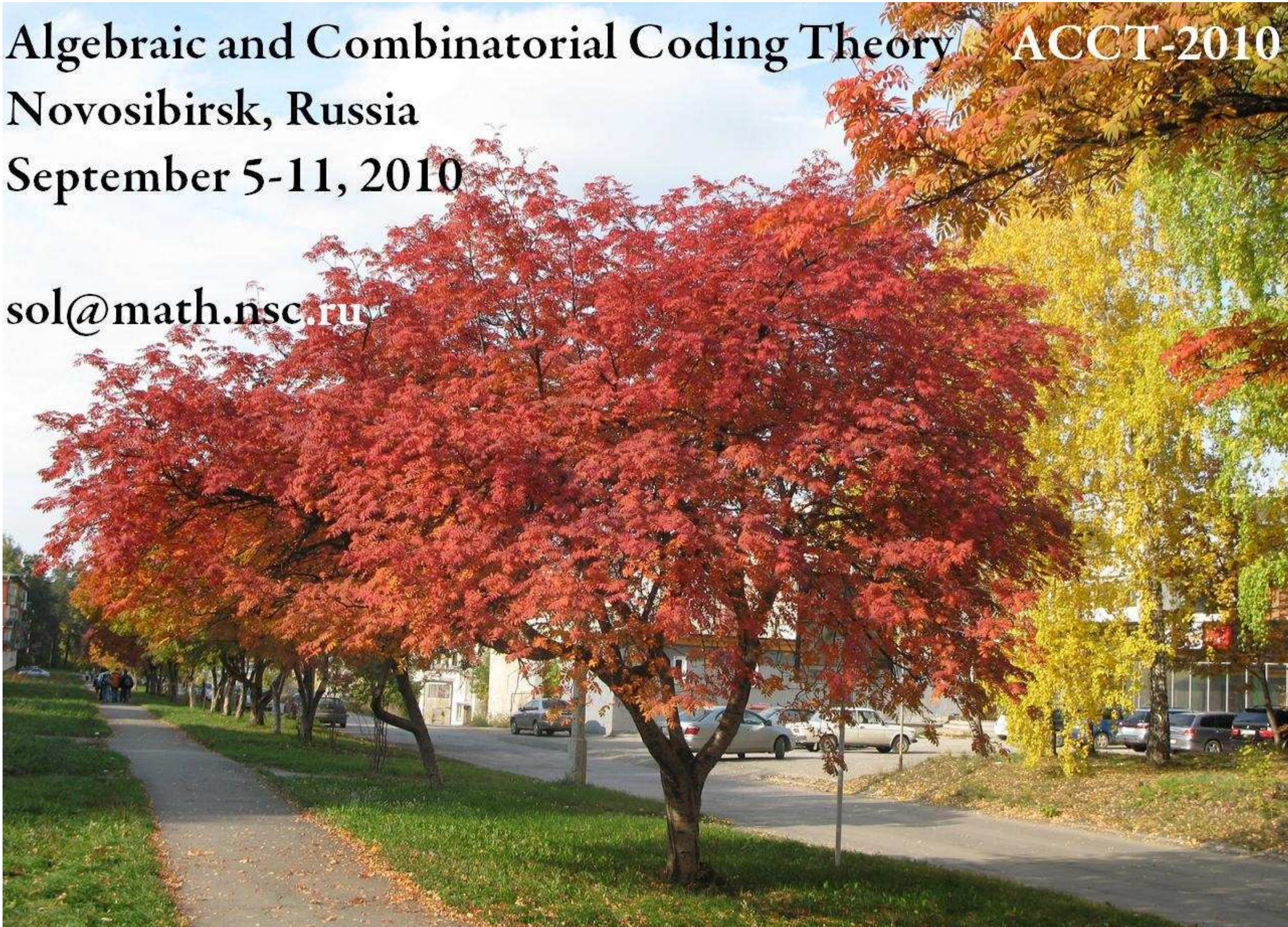
Hamming graph: $\Pi_w(\cdot) = K_w(K_w^{-1}(\cdot))$, Krawtchouk polynomials.

$$K_w(z) = K_w(z; n, q) = \sum_{j=0}^{w} (-1)^j (q-1)^{w-j} \binom{z}{j} \binom{n-z}{w-j}$$

Johnson graph: $\Pi_w(\cdot) = E_w(E_w^{-1}(\cdot))$, Eberlein polynomials.

Conclusions

We have a universal matrix formula for calculating the weight distribution of a perfect coloring with respect to a completely regular code. In the case of weight distribution with respect to a point in a Hamming or Johnson graph, an explicit form of this formula is given, using Krawtchouk or Eberlein polynomials.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