On ± 1-error correctable integer residue codes

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Introduction

Coded modulation is the collective term for all techniques which combine and jointly optimize channel coding and modulation for digital transmission.

- **Trellis coded modulation (TCM)**: It consists in an expanding the input bits by a binary convolutional code and partitioning the used signal constellation into smaller subsets with a larger intra-set distance.
- **Integer coded modulation (ICM)**: A type of block coded modulation - each point of the signal constellation corresponds to a symbol of $\mathbb{Z}_A$ and coded by a code over $\mathbb{Z}_A$.
- **Others**: Coded modulation based on Gaussian and algebraic integers.

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Integer codes

Definition

Let $\mathbb{Z}_A$ be the ring of integers modulo A and $\mathbf{H}$ be an $m \times n$ matrix with entries in $\mathbb{Z}_A$. An *integer code* over $\mathbb{Z}_A$ of length n with a parity-check matrix $\mathbf{H}$ is a subset of $\mathbb{Z}_A^n$, defined by

$$\mathcal{C} = \mathcal{C}(\mathbf{H}, \mathbf{d}) = \{\mathbf{c} \in \mathbb{Z}_A^n \mid \mathbf{c}\mathbf{H}^T = \mathbf{d} \mod A\}$$

where $\mathbf{d} \in \mathbb{Z}_A^m$. Usually $\mathbf{d}$ is the all-zero vector and then we say that $\mathcal{C}$ is an $[n, n-m]$ code.

The integer codes were introduced by R. Varshamov and Tenengolz (1965) in order to correct a single insertion/deletion error per codeword.

Integer codes

- **R. Varshamov and Tenengolz (1965)**: Ins/Del errors
- **Blake 1972-75**
- **Spigel 1977**
- **Nakamura 1977**: DPSK
- **Nilsson 1993**: PSK
- **Levenshtein and Vink 1993, U. Tamm 1998**: peak shift errors
- **Baldini and Farrell 1994**
- **Vink and Morita 1998**: PSK, synch
- **Kostadinov, Morita Manev 2003-04**: QAM

Error correcting capability

When a codeword $\mathbf{c} \in \mathcal{C}$ is sent through a noisy channel the received vector can be written in the form

$$\mathbf{r} = \mathbf{c} + \mathbf{e}, \quad \mathbf{e} = (e_1, \ldots, e_n) \in \mathbb{Z}_A^n.$$

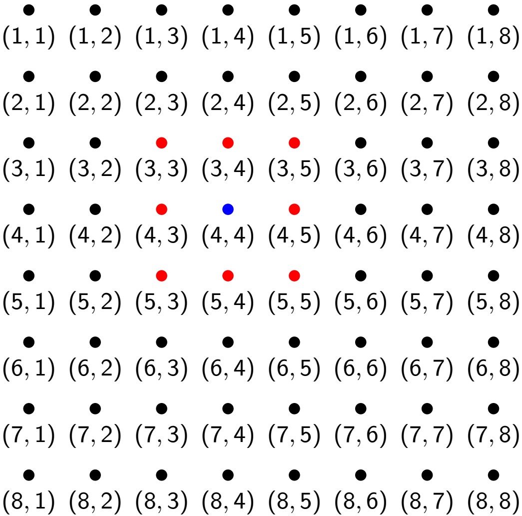
we say that t errors occurred in $\mathbf{c}$, if t of the entries of $\mathbf{e}$ are nonzero.

Definition

Let C be an $[n, k]$ code over the integer ring $\mathbb{Z}_A$. C is a t-multiple $(\pm e_1, \pm e_2, \ldots, \pm e_s)$-error correctable code if it can correct (up to) any t errors with values from the set $\{\pm e_1, \pm e_2, \ldots, \pm e_s\}$, which are occurred in a codeword.

Why (± 1)-error correctable codes?

Let us consider square M-QAM constellation with $M = 2^{2k}$. Let us label each signal point in M-QAM constellation by a pair $(i, j) \in \mathbb{Z}_A \times \mathbb{Z}_A$ of elements of $\mathbb{Z}_A$ where $A \geq 2^k$.



Bounds on the size of alphabet

Proposition

If $\mathcal{C}$ correct two errors of type $(\pm e_1, \pm e_2, \ldots, \pm e_s)$ then the cardinality, A, of the ring satisfies the inequality

$$A^{n-k} \geq 2sn(2sn - n + 1) + 1.$$

In particular if $\mathcal{C}$ is a double ± 1-error correctable code, then

$$A \geq 2n^2 + 1; \qquad \textit{when } k = n - 1 \tag{1}$$

$$A \geq \sqrt{2n^2 + 1} \qquad \textit{when } k = n - 2. \tag{2}$$

Single ± 1-errors correction

Theorem

Let $l > 1$ be an integer. For every $n \geq 2^{l-1}$ there exists a (± 1) single error correctable code of length n over $\mathbb{Z}_{2^l}$ with an $m \times n$ check matrix,

$$\mathbf{H} = \left(\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_i, \ldots, \mathbf{h}_n\right),$$

where m is defined by

$$2^{m-2}\left(2^{(m-1)(l-1)} - 1\right) < n \leq 2^{m-1}\left(2^{m(l-1)} - 1\right)$$

and every column $\mathbf{h}_i$ belongs to

$$S^1 = \{(s_1, s_2, \ldots, s_m)^\tau \mid \; s_1 \in \mathbb{Z}^*_{2^{l-1}},\; s_i \in \mathbb{Z}_{2^{l-1}}, i = 2, \ldots, m\},$$

or to

$$S^2 = \{(s_1, s_2, \ldots, s_m)^\tau \mid \; s_1 \in \{0, 2^{l-1}\},\; s_i \in \mathbb{Z}^*_{2^{l-1}}, i = 2, \ldots, m\}.$$

Double ± 1-error correctable codes

Proposition

Up to equivalence the parity check matrix of an $[n, n-2]$ double ± 1-error correctable code over $\mathbb{Z}_A$ has the form

$$\mathbf{H} = \begin{pmatrix} 1 & 0 & h_{13} & \dots & h_{1n} \\ 0 & 1 & h_{23} & \dots & h_{2n} \end{pmatrix} \quad \text{or} \quad \mathbf{H} = \begin{pmatrix} 1 & h_{12} & h_{13} & \dots & h_{1n} \\ 0 & a & h_{23} & \dots & h_{2n} \end{pmatrix},$$

where $a \mid A$, $a > 1$.

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 2 & 3 & \dots & n-1 \\ 1 & 0 & h_{23} & h_{24} & \dots & h_{2n} \end{pmatrix}$$

over a ring $\mathbb{Z}_A$ with $A \geq 2n-1$ is at least single ± 1-error correctable code.

Examples

- $[6,4]$ code over $\mathbb{Z}_{16}$ with

$$\mathbf{H} = \begin{pmatrix} 5 & 4 & 3 & 2 & 1 & 0 \\ 12 & 6 & 3 & 5 & 0 & 1 \end{pmatrix}$$

- $[8,6]$ code over $\mathbb{Z}_{16}$ with

$$\mathbf{H} = \begin{pmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 7 & 0 & 12 & 6 & 3 & 5 & 0 & 1 \end{pmatrix}$$

- $[8,6]$ code over $\mathbb{Z}_{17}$ with

$$\mathbf{H} = \begin{pmatrix} 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 7 & 0 & 12 & 6 & 3 & 5 & 0 & 1 \end{pmatrix}$$

Decoding

Let a sequence of signal points, $s_{i_1 j_1}, s_{i_2 j_2}, \ldots, s_{i_n j_n}$, be sent trough the channel. In the coded case $(i_1, i_2, \ldots, i_n)$ and $(j_1, j_2, \ldots, j_n)$ are codewords. At the receiver the decoder based on the received signal sequence $r_{i_1 j_1}, r_{i_2 j_2}, \ldots, r_{i_n j_n}$, outputs a sequence of signal points $s'_{i_1 j_1}, s'_{i_2 j_2}, \ldots, s'_{i_n j_n}$. Let q_u and q_b be the probabilities for correct demodulations in uncoded and coded cases respectively.

$$q_u = (1 + 15\mathrm{erf}(\gamma))^2/256,$$
$$q_c = (3 + 13\mathrm{erf}(3\gamma))/256,$$

where $\gamma = \sqrt{E_s/170N_0}$ and $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$.

$$P_{SE}(\mathcal{C}) = \frac{1}{n}\left(1 - q_u^n - nq_u^{n-1}q_c - \binom{n}{2} q_u^{n-2} q_c^2\right)$$

Decoding

- **Hard decoding**: If a syndrome of the received vector does not belong to the list of possible syndromes the decoder leaves the values (on the corresponding axis) unchanged.
- **Soft decoding**: The classical soft decoding for "big square"(i.e., there are 9 possible values for each signal point).
- **Mixed decoding**: The decoder applies soft decoding when the syndromes are not among the possible ones.

Simulation results

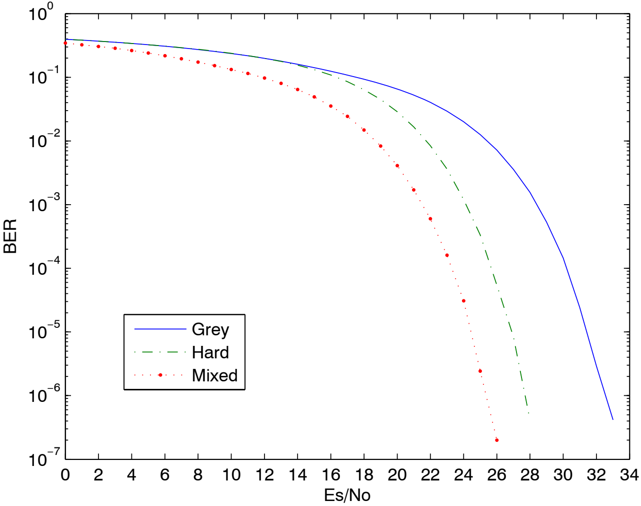


Figure: 256-QAM: Grey, hard, and mixed decoding $[6,4]$ code over $\mathbb{Z}_{16}$.

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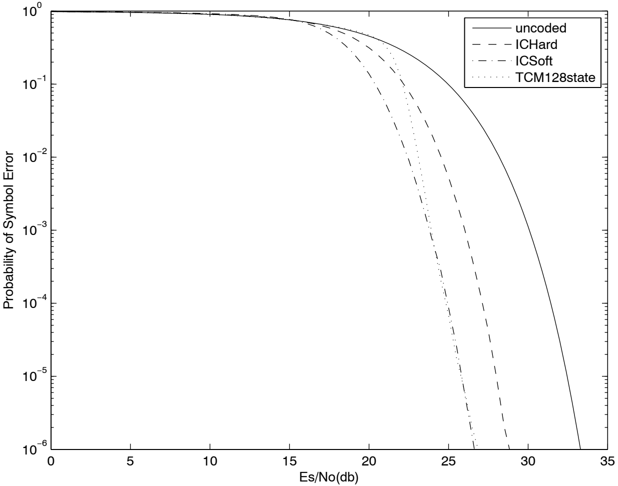


Figure: 256-QAM: $[8, 6]$ code over $\mathbb{Z}_{19}$.

Simulation results

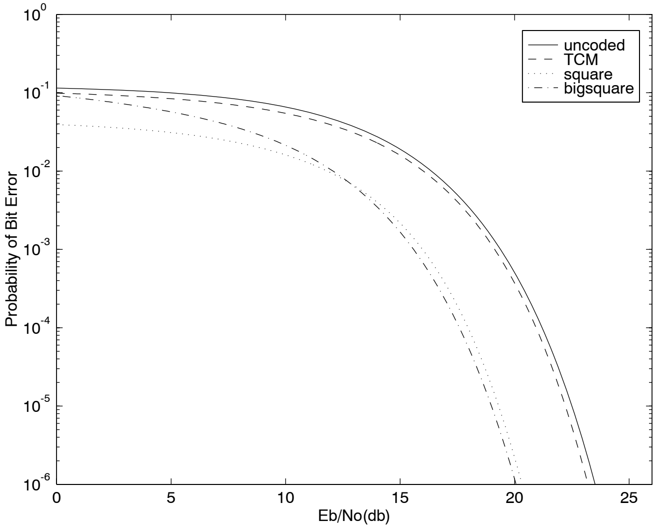
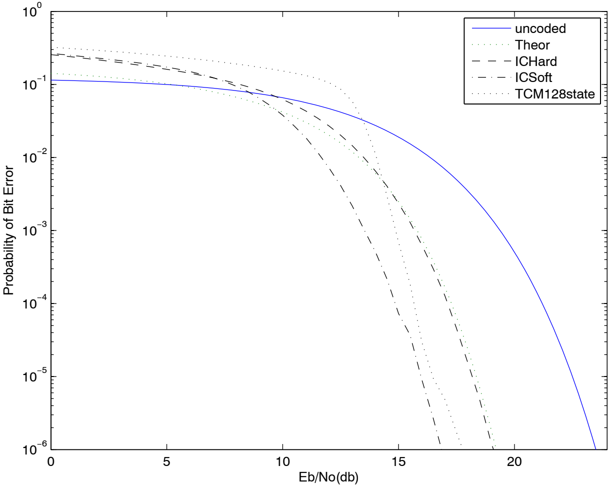


Figure: 256-QAM: $[8, 6]$ code over $\mathbb{Z}_{19}$.

Simulation results



Conclusions

- Simple realization and good performance
- Can be used as an inner code in cascading schemes
- Further research on their performance when are used in combinations with other coding schemes

THANK YOU

FOR ATTENTION