Good (n,r)-arcs in PG(2,23)

Multiple blocking sets in PG(2,23)

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Introduction

Let **GF(q)** denote the **Galois field** of q elements

Let **V(3,q)** be the **vector space** of row vectors of length three with entries in GF(q).

Let **PG(2,q)** be the corresponding **projective plane**.

The **points** (x_1, x_2, x_3) of PG(2,q) are the 1-dimensional subspaces of V(3,q).

Subspaces of dimension two are called **lines**.

Introduction

The number of $k-$dimensional subspaces of V(n,q) is

$$\frac{(q^n-1)(q^n-q)\dots(q^n-q^{k-1})}{(q^k-1)(q^k-q)\dots(q^k-q^{k-1})}$$

The **number of points** is q^2+q+1 $(\ n=3, k=1)$

The **number of lines** is q^2+q+1 $(n=3, k=2)$

There are $q+1$ points on every line and $q+1$ lines through every point.

Introduction - (n, r)-arc

Definition: *An* (n, r)*-arc is a set of* n *points of a projective plane such that some* r , *but no* $r + 1$ *of them, are collinear.*

Notation: $m_r(2, q) \rightarrow$ *the maximum size of an (n,r)-arc*

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Note that an (n, r) -arc is the complement of a $(q^2 + q + 1 - n, q + 1 - r)$-blocking set in a projective plane and conversely.

Introduction - *secant distribution*

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Basic results in PG(2,q)

Bose (1947) proved that

$$m_2(2,q) = q+1 \qquad \text{for } q \text{ - odd}$$

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From the results of **Barlotti** (1965) and **Ball** (1994) it follows that

$$m_r(2,q) = (r-1)q+1$$

for q odd prime and

$$r = (q+1)/2, \quad r = (q+3)/2$$

Barlotti's construction - PG(2,23)

Let C be **a conic** = (q+1,2) arc = (24,2)-arc

0-secants (**exterior lines**) - q(q-1)/2 = 253

1-secants (**tangents**) - q+1 = 24

2-secants (**secants**) - (q+1)q/2 = 276

Barlotti's construction - PG(2,23)

$q^2 + q + 1 =$ **553** points of PG(2,23)

There are **three types of points** with respect to their position relative to C (**a conic**)

- **276 exterior points** - *on precisely 2 tangents*
- **253 interior points** - *on no tangents at all*
- **24 points of C**

Barlotti's construction- PG(2,23)

All interior points plus **one point of C** $\rightarrow$ (254,12) arc

$$m_{12}(2, 23) = 254$$

Barlotti's construction- PG(2,23)

All interior points plus **one point of C** $\rightarrow$ (254,12) arc

$$m_{12}(2,23) = 254$$

All interior points plus **all points of C** $\rightarrow$ (277,13) arc

$$m_{13}(2,23) = 277$$

Additional interesting facts

C - *a conic*. **Lines -** *secants; tangents; exterior lines*

Every secant has $\frac{q-1}{2}$ exterior points and $\frac{q-1}{2}$ interior points

Each tangent has q exterior points and no interior points

Each exterior line has $\frac{q+1}{2}$ exterior points and $\frac{q+1}{2}$ interior points

Upper bounds on $m_r(2, q)$

Theorem 1: (Ball, 1996)

Let K be an (n, r)-arc in $\mathrm{PG}(2,\ \mathrm{q})$ *where q is prime.*

If $r \leq \frac{q+1}{2}$ *then* $\quad m_r(2, q) \leq (r-1)q + 1$

If $r \geq \frac{q+3}{2}$ *then* $m_r(2, q) \leq (r-1)q + r - (q+1)/2$

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Theorem 2: (Daskalov, OC'2005)

Let K be an (n, r)-arc in $\mathrm{PG}(2,\ \mathrm{q})$, $\quad q \le 29$ *is prime.*
If $r > \frac{q+3}{2}$ $\quad$ *then*

$$m_r(2, q) \le (r - 1)q + r - (q + 3)/2$$

$m_r(2,q)$ - known values

r\q	3	4	5	7	8	9
2	4	6	6	8	10	10
3		9	11	15	15	17
4			16	22	28	28
5				29	33	37
6				36	42	48
7					49	55
8						65

$m_r(2, q)$ - bounds

r\q	11	13	17	19
2	12	14	18	20
3	21	23	28..35	31..39
4	32..34	38..40	48..52	52..58
5	43..45	49..53	61..69	68..77
6	56	64..66	78..86	86..96
7	67	79	94..103	105..115
8	77..78	92	114..120	124..134
9	89..90	105	137	147..153
10	100..102	118..119	154	172

$m_r(2, q)$ - bounds

r\q	11	13	17	19
11		132..133	166..171	191
12		145..147	182..189	204..210
13			204..207	225..230
14			221..225	242..250
15			239..243	262..270
16			256..261	285..290
17				305..310
18				324..330

Recent results on $m_r(2, q)$

- S. Ball **q = 11, 13**
- Marcugini, Milani, Pambianco **q = 11, 13**
- Daskalov, Contreras **q = 13**
- Daskalov, Metodieva **q = 17**
- Braun, Kohnert, Wassermann **q = 11, 13, 16, 17, 19**

Arcs in PG(2,23)

***There exist*:**

1. A (35,3)- arc

2. A (58,4) - arc

3. A (77,5) - arc

4. A (97,6) - arc

5. A (119,7) - arc

Blocking sets in PG(2,23)

There exist:

1. **A (193,7)- blocking set**
2. **A (168,6)- blocking set**
3. **A (142,5)- blocking set**
4. **A (116,4)- blocking set**
5. **A (92,3)- blocking set**
6. **A (69,2)- blocking set**

Arcs in PG(2,23)

There exist:

6. A (360,17) - arc $\Leftrightarrow$ a (193,7)- blocking set

7. A (485,18) - arc $\Leftrightarrow$ a (168,6)- blocking set

8. A (411,19) - arc $\Leftrightarrow$ a (142,5)- blocking set

9. A (437,20) - arc $\Leftrightarrow$ a (116,4)- blocking set

10. A (461,21) - arc $\Leftrightarrow$ a (92,3)- blocking set

11. A (484,22) - arc $\Leftrightarrow$ a (69,2)- blocking set

A (35,3)-arc in PG(2,23)

$$\begin{array}{ccccc}
(0,0,1), & (0,1,0), & (0,1,1), & (1,0,0), & (1,0,1), \\
(1,1,0), & (1,1,2), & (1,3,g), & (1,3,h), & (1,4,1), \\
(1,5,8), & (1,5,f), & (1,7,4), & (1,7,k), & (1,8,d), \\
(1,8,m), & (1,9,8), & (1,a,6), & (1,a,h), & (1,b,9), \\
(1,b,e), & (1,e,3), & (1,e,l), & (1,f,a), & (1,f,d), \\
(1,h,b), & (1,h,c), & (1,k,2), & (1,k,m), & (1,l,7), \\
(1,l,g), & (1,m,5), & (1,m,i), & (1,n,1), & (1,n,n)
\end{array}$$

$$\tau_0 = 172,\ \tau_1 = 58,\ \tau_2 = 187,\ \tau_3 = 136$$

$a = 10, b = 11, c = 12, d = 13, e = 14, f = 15, g = 16,$
$h = 17, i = 18, k = 19, l = 20, m = 21, n = 22$

The new results - table

r\q	11	13	17	19	23
2	12	14	18	20	24
3	21	23	28..35	31..39	35..47
4	32..34	38..40	48..52	52..58	58..70
5	43..45	49..53	61..69	68..77	77.. 93
6	56	64..66	78..86	86..96	97..116
7	67	79	94..103	105..115	119..139
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10	100..102	118..119	154	172	***..208
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r\q	11	13	17	19	23
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14			221..225	242..250	***..300
15			239..243	262..270	***..324
16			256..261	285..290	***..348
17				305..310	360..372
18				324..330	385..396
19					411..420
20					437..444
21					461..468
22					484..492