Refined Upper Bounds for Ring-Linear Codes

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Outline

- Codes over finite fields
- Code optimality
- Bounds for codes for the Hamming weight
- Ring-linear coding
- The homogeneous weight
- Bounds on the size of a code for the homogeneous weight

Notation

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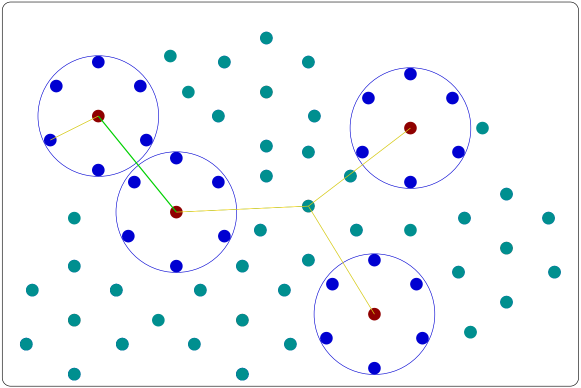
- $F = GF(q)$, $q = p^m$ some prime p
- R is a finite ring with identity
- $\hat{R} := \mathrm{Hom}_{\mathbf{Z}}(R, \mathbb{C}^\times)$ the characters on $(R, +)$
- $\chi \in \hat{R}$ is a character on $(R, +)$
- C is a code of length n and minimum distance d

Sphere Packing

A good code is one with a large number of codewords for a given minimum distance and length.



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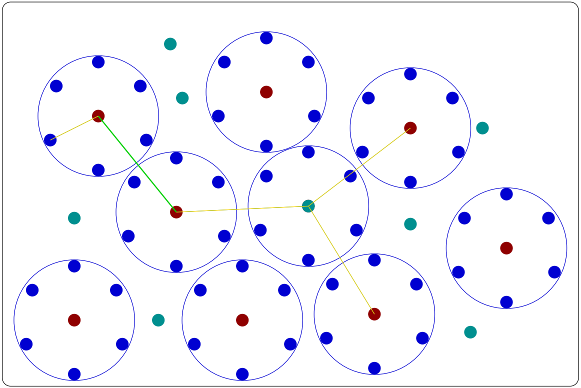

Code Optimality

The Main Coding Problem:

1. For fixed length n and minimum distance d, what is the maximum size of any code over R?
   i.e., what is $A_R(n, d)$?

2. For a fixed length n and minimum distance d, what is the maximum size of any linear code over R?
   i.e., what is $B_R(n, d)$?

Some Distance Functions

Definition (Hamming Metric)

Let $\mathbf{u}, \mathbf{v} \in R^n$. The Hamming distance between $\mathbf{u}$ and $\mathbf{v}$ is the number of components where $\mathbf{u}$ and $\mathbf{v}$ differ, i.e.

$$\mathrm{d}_{\mathrm{Ham}}(\mathbf{u}, \mathbf{v}) = |\{i : u_i \neq v_i\}|$$

$$\mathbf{u} = [0, 0, 1, 1, 3, 3], \mathbf{v} = [1, 2, 2, 1, 1, 3] \in \mathbf{Z}_4$$

$$d_{\mathrm{Ham}}(\mathbf{u}, \mathbf{v}) = 4.$$

Some Distance Functions

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Definition (Lee Metric)

Let $u, v \in \mathbf{Z}_m$. The Lee distance between u and v is the absolute value modulo m of $u - v$, i.e.

$$\mathrm{d}_{\mathrm{Lee}}(u, v) = |u - v|_m = \begin{cases} u - v & \text{if } u - v \in \{0, \ldots, \lfloor m/2 \rfloor\} \\ v - u & \text{otherwise} \end{cases}$$

If $\mathbf{u}, \mathbf{v} \in \mathbf{Z}_m^n$ then $\mathrm{d}_{\mathrm{Lee}}(\mathbf{u}, \mathbf{v}) = \sum_{i=1..n} |u_i - v_i|_m$.

$$\mathbf{u} = [0, 0, 1, 1, 3, 3], \mathbf{v} = [1, 2, 2, 1, 1, 3] \in \mathbf{Z}_4$$

$$d_{\mathrm{Lee}}(\mathbf{u}, \mathbf{v}) = 1 + 2 + 1 + 2 = 6$$

Some Bounds for Codes over Finite Fields

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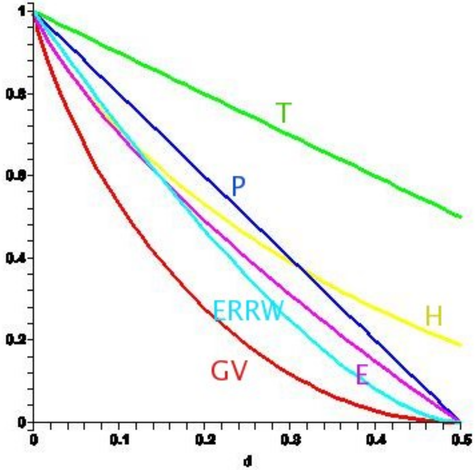
- Singleton: $|C| \leq A_q(n,d) \leq q^{n-d+1}$
- Hamming: $|C| \leq A_q(n,d) \leq \frac{q^n}{V_q(n,\lfloor \frac{d-1}{2} \rfloor)}$,
- Plotkin: $|C| \leq A_q(n,d) \leq \frac{d}{d-\gamma n}, \gamma = \frac{q-1}{q}$, if $n < \frac{d}{\gamma}$
- Gilbert-Varshamov: $A_q(n,d) \geq \frac{q^n}{V_q(n,d-1)}$
- Elias-Bassalygo bound
- Mc-Eliece-Rodemich-Rumsey-Welch bound
- Linear Programming bound

Asymptotic Representations



Codes over Finite Rings

Definition

An code of length n over R is a nonempty subset of R^n. A (left) linear code of length n over R is a left R-submodule of R^n.

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We will usually assume that R is a finite Frobenius ring.

Many of the foundational results of classical coding theory (e.g. the MacWilliams' theorems) can be extended to the finite ring case when R is Frobenius.

[Wood, Honold, Nechaev, Greferath, Schmidt..]

Finite Frobenius Rings

For a finite ring R, $\hat{R}$ is an $R-R$ bimodule via

$$\chi^r(x) = \chi(rx), \quad {}^r\chi(x) = \chi(xr)$$

for all $x, r \in R, \chi \in \hat{R}$.

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R is a finite Frobenius ring iff

$$\begin{gathered} soc\ {}_RR \text{ is left principal, iff} \\ {}_R(R/rad\ R) \simeq\ soc\ \ {}_RR\text{, iff} \\ {}_RR \simeq\ {}_R\hat{R} \end{gathered}$$

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Then ${}_R\hat{R} = {}_R\langle\chi\rangle$ for some (left) generating character χ.

Finite Frobenius Rings

Let R and S be finite Frobenius rings, let G be a finite group. The following are examples of Frobenius rings.

- integer residue rings $\mathbf{Z}_m$
- Galois rings
- principal ideal rings
- $R \times S$
- the matrix ring $M_n(R)$
- the group ring $R[G]$

Not a Frobenius Ring

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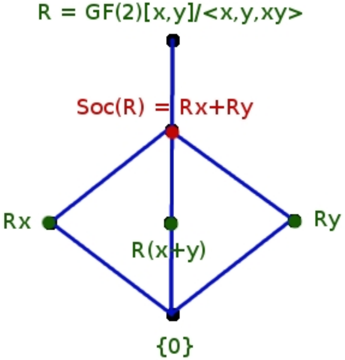

$R = \mathbb{F}_2[x,y]/\langle x,y,xy\rangle$ is not Frobenius since $R/Rad\ R = R$, but $Soc\ R = Rx + Ry$.

Homogeneous Weights

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Definition

A weight $w : R \longrightarrow \mathbf{Q}$ is *(left) homogeneous*, if $w(0) = 0$ and

1. If $Rx = Ry$ then $w(x) = w(y)$ for all $x, y \in R$.
2. There exists a real number γ such that

$$\sum_{y \in Rx} w(y) \;=\; \gamma\, |Rx| \qquad \text{for all } x \in R \setminus \{0\}.$$

Examples of Homogeneous Weights

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Example

On every finite field $\mathbb{F}_q$ the Hamming weight is a homogeneous weight of average value $\gamma = \frac{q-1}{q}$.

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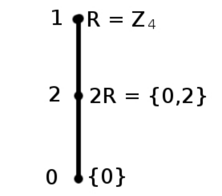
Example

On $\mathbf{Z}_4$ the Lee weight is homogeneous with $\gamma = 1$.

x	0	1	2	3
$w_{\mathrm{Lee}}(x)$	0	1	2	1

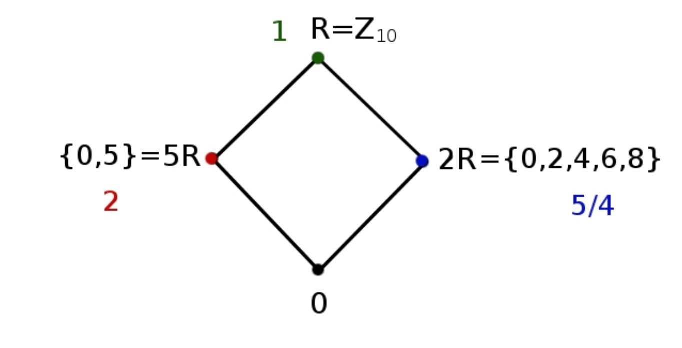


Examples of Homogeneous Weights

Example

On $\mathbf{Z}_{10}$ the following weight is homogeneous with $\gamma = 1$:

x	0	1	2	3	4	5	6	7	8	9
$\mathrm{w}_{\mathrm{hom}}(x)$	0	1	$\frac{5}{4}$	1	$\frac{5}{4}$	2	$\frac{5}{4}$	1	$\frac{5}{4}$	1



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Example

On the ring R of 2×2 matrices over GF(2) the weight

$$w : R \longrightarrow \mathbb{R}, \quad X \mapsto \begin{cases} 0 & : \quad X = 0, \\ 2 & : \quad X \text{ singular}, \ X \neq 0, \\ 1 & : \quad \text{otherwise}, \end{cases}$$

is a homogeneous weight of average value $\gamma = \frac{3}{2}$.

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R has 3 principal ideals - each of size - 4 and six units. We have

$$0 + \underbrace{9\frac{4}{3}\gamma}_{\text{sing.}} + \underbrace{6\frac{2}{3}\gamma}_{\text{units}} = 16\gamma.$$

Examples of Homogeneous Weights

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Example

On a local Frobenius ring R with q-element residue field the weight

$$w : R \longrightarrow \mathbb{R}, \quad x \mapsto \begin{cases} 0 & : \quad x = 0, \\ \frac{q}{q-1} & : \quad x \in soc(R),\ x \neq 0, \\ 1 & : \quad \text{otherwise}, \end{cases}$$

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Which finite rings admit a homogeneous weight?

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Which finite rings admit a homogeneous weight?

Up to the choice of γ, every finite ring admits a unique homogeneous weight .

Homogeneous Weights of FFRs

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Theorem (Honold)

Let R be a finite Frobenius ring with generating character χ. Then the homogeneous weights on R are precisely the functions

$$w : R \longrightarrow \mathbb{R}, \quad x \mapsto \gamma\Big[1 - \frac{1}{|R^\times|}\sum_{u \in R^\times} \chi(xu)\Big]$$

where γ is a real number.

Bounds on $A_R(n,d)$ for the Homogeneous Weight

The following bounds have been found for codes over FFRs for the homogeneous weight.

- Sphere-packing (Hamming)
- Sphere-covering (Gilbert-Varshamov)
- Plotkin-like bounds
- Elias-like bounds
- Singleton-like bound
- Linear programming bound

A Key Lemma

Definition

Let $x \in R^n$, $C <_R R^n$. We define $\pi_C(x) := (x_i)_{i \notin \mathrm{supp} C}$.

Lemma

Let $C \leq {}_R R^n$ be a linear code, and let $x \in R^n$. Then

$$\frac{1}{|C|} \sum_{c \in C} w(x + c) = \gamma |\mathrm{supp} C| + w(\pi_C(x)).$$

A Key Lemma

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Proof: WLOG, let $\gamma = 1$. We compute $\sum_{c \in C} w(x+c)$

$$
\begin{aligned}
&= \sum_{c \in C} \sum_{i \in \operatorname{supp} C} w(x_i + c_i) + \sum_{c \in C} \sum_{i \notin \operatorname{supp} C} w(x_i) \\
&= \sum_{i \in \operatorname{supp} C} \sum_{c \in C} w(x_i + c_i) + |C| w(\pi_C(x)) \\
&= \sum_{i \in \operatorname{supp} C} \sum_{c \in C} \left[1 - \frac{1}{|R^\times|} \sum_{u \in R^\times} \chi(u(x_i + c_i)) \right] + |C| w(\pi_C(x)) \\
&= |\operatorname{supp} C| \left[|C| - \frac{1}{|R^\times|} \sum_{u \in R^\times} \chi(ux_i) \sum_{c \in C} \chi(uc_i) \right] + |C| w(\pi_C(x \\
&= |C| [|\operatorname{supp} C| + w(\pi_C(x))].
\end{aligned}
$$

Residual Codes

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Definition

Let $C \leq_R R^n, c \in R^n$. $Res(C, c) := \{(x_i) : x \in C, c_i = 0\}$.

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Example

Let C be the $\mathbb{Z}_4$-linear code generated by

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 3 & 1 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 2 & 3 & 1 \\ 0 & 0 & 1 & 0 & 3 & 3 & 3 & 2 \\ 0 & 0 & 0 & 1 & 2 & 3 & 1 & 1 \end{bmatrix}.$$

Let $c = [0, 0, 0, 2, 0, 2, 2, 2]$. Then $Res(C, c)$ is generated by

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Residual Codes - the Main Theorem

Theorem (BGKS)

Let $C \leq {}_R R^n$ have minimum homogeneous weight d, and let $c \in C$ satisfy $\ell(c) := w_{\mathrm{Ham}}(c) < \frac{d}{\gamma}$. Then $Res(C, c)$ has

- *length $n - \ell(c)$,*
- *minimum homogeneous weight $d' \geq d - \gamma\ell(c)$,*
- *$Res(C, c) \cong C/Rc$: in particular $|Res(C, c)| = \dfrac{|C|}{|Rc|}$,*
- $$|C| \;\leq\; |Rc| \, \frac{d - \gamma\ell(c)}{d - \gamma n}.$$

Bounds on $B_R(n,d)$ for the Homogeneous Weight

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Corollary (BGKS)

Let $C \leq {}_R R^n$ be a linear code of minimum homogeneous weight d and minimum Hamming weight ℓ where $\ell \leq n < \frac{d}{\gamma}$. Then

$$|C| \leq |R| \frac{d - \gamma\ell}{d - \gamma n}.$$

Bounds on $B_R(n, d)$ for the Homogeneous Weight

Corollary (BGKS)

Let $C \leq {}_R R^n$ be a linear code of minimum homogeneous weight d and minimum Hamming weight ℓ where $\ell < n < \frac{d}{\gamma}$. Let Q be the maximum size of any minimal ideal of R. Then

$$|C| \leq Q \, \frac{d - \gamma\ell}{d - \gamma n}.$$

A Plotkin Optimal Code

Example

Let $R = \mathbb{F}_2^{2\times 2}$. Let C be the length $16^m - 1$ Simplex Code over R. Then $|C| = |R|^m = 16^m$,

$$d = |R|^m \gamma = 16^m \gamma,$$

$$\ell := \mathrm{d}_{\mathrm{Ham}}(C) = 16^m - \frac{16^m}{4} = \frac{3}{4} 16^m.$$

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R has 3 minimal ideals, each of size $Q = 4$ and so

$$\begin{aligned} |C| &\leq Q\,\frac{d - \gamma\ell}{d - \gamma n} \\ &= 4\,\frac{16^m\gamma - \frac{3}{4}16^m\gamma}{16^m\gamma - (16^m - 1)\gamma} = 4\,\frac{16^m}{4} = 16^m. \end{aligned}$$

Bounds on $B_R(n,d)$ for the Homogeneous Weight

Singleton-like bounds:

Theorem (BGKS)

Let $C \leq {}_R R^n$ be an $[n,d]$ linear code and suppose that $n \leq \frac{d}{\gamma}$. Then

$$n - \left\lceil \frac{|R|-1}{|R|} \frac{d}{\gamma} \right\rceil \geq \left\lceil \log_{|R|} |C| - 1 \right\rceil .$$

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Theorem (BGKS)

Let C be an $[n,d]$ code over R satisfying $n \leq \frac{d}{\gamma}$ and $\ell(C) < n$. Let $P := \max\{|Ra| : a \in R^n, Ra \leq C, \ell(a) < n\}$. Then

$$n - \left\lceil \frac{P-1}{P} \frac{d}{\gamma} \right\rceil \geq \lceil \log_P |C| - \log_P |R| \rceil .$$

Singleton-Like Bounds

These bounds arise by repeated applications of the main theorem, to obtain a sequence of $[n_i, d_i]$ codes

$$C = C_0,\ C_1 \cong C/Rc,\ C_2 \cong C_1/Rc^1,\ \ldots,\ C_r \cong C_{r-1}/Rc^{r-1}$$

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$$\begin{aligned} |C| &= |Rc^0||Rc^1|\cdots|Rc^{r-1}||C_r| \\ &\leq P^{r-1}|C_r| \leq P^{r-1}|R| \end{aligned}$$

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where $P = \max\{|Ra| : a \in R^n, Ra \leq C, \ell(a) < n\}$, so

$$n \geq \frac{|Rc|-1}{|Rc|}\frac{d}{\gamma} + r \geq \frac{P-1}{P}\frac{d}{\gamma} + \log_P(|C| - |R|).$$

A Class of MDS Codes

Example

Let R be a chain ring of length 2. Then $R^{\times} = R \backslash rad\ R$ and $|R| = q^2$. Let $U := R^2 \backslash rad\ R^2$, let $\mathcal{P} := \{xR : x \in U\}$. Then $|\mathcal{P}| = q^2 + q$.

Let $C <_R R^n$ be the length $n := q^2 + q$ code with $2 \times n$ generator matrix whose columns are the distinct elements of $\mathcal{P}$.

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Example (cont.)

Setting $\gamma = 1$, each nonzero word xG of C has weight

$$w(xG) = \begin{cases} q^2 + q & \text{if } x \in U \\ \frac{q^3}{q-1} & \text{if } x \in radR^2 \backslash 0 \end{cases},$$

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$$\begin{aligned} \Longrightarrow \; n - \left\lceil \frac{P-1}{P} d \right\rceil &= n - \left\lceil \frac{q^2-1}{q^2}(q^2+q) \right\rceil \\ &= n - \left\lceil q^2 + q - 1 - \frac{1}{q} \right\rceil \\ &= q^2 + q - q^2 - q + 1 = 1 = r. \end{aligned}$$

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