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New asymptotic bounds for some spherical $(2k-1)$-designs

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Abstract. We apply polynomial techniques to investigate the structure of spherical designs in an asymptotic process with fixed odd strength while the dimension and odd cardinality tend to infinity in certain relation. This gives new lower bounds on the quantity $B_{\rm odd}(n,\tau)=\min\{|C|: C\subset \mathbb{S}^{n-1}$ is a τ-design with odd cardinality $|C|\}$.

1 Introduction

The spherical designs were introduced in 1977 by Delsarte-Goethals-Seidel [8]. A spherical τ-design $C\subset\mathbb{S}^{n-1}$ is a finite nonempty subset of $\mathbb{S}^{n-1}$ such that

$$\frac{1}{\mu(\mathbb{S}^{n-1})}\int_{\mathbb{S}^{n-1}} f(x)d\mu(x)=\frac{1}{|C|}\sum_{x\in C} f(x)$$

holds for all polynomials $f(x)=f(x_1,x_2,\ldots,x_n)$ of degree at most τ. If $C\subset\mathbb{S}^{n-1}$ is a spherical τ-design and $x\in C$ then (cf. [9])

$$\sum_{y\in C\setminus\{x\}} f(\langle x,y\rangle)=f_0|C|-f(1)$$

holds, where f_0 is the first coefficient in the expansion of $f(t)=\sum_{i=0}^k f_iP_i^{(n)}(t)$ in terms of the Gegenbauer polynomials [1, Chapter 22].

Delsarte-Goethals-Seidel [8] proved that if $C\subset\mathbb{S}^{n-1}$ is a spherical τ-design, then

$$|C|\geq\begin{cases} 2\binom{n+k-2}{n-1}, & \text{if } \tau=2k-1,\\ \binom{n+k-1}{n-1}+\binom{n+k-2}{n-1}, & \text{if } \tau=2k.\end{cases}$$

We denote $B_{\mathrm{odd}}(n,\tau) = \min\{|C| : C \subset \mathbb{S}^{n-1}$ is a τ-design with odd cardinality $|C|\}$ and consider the following problem.

Problem. For fixed integer $\tau = 2k-1 \geq 3$ and for $n \to \infty$ obtain lower bounds for $B_{\mathrm{odd}}(n,\tau)$.

We extend a method for proving nonexistence of spherical $(2k-1)$-designs with odd cardinality $|C| = M$ which was proposed recently in [4, 5] to work in our problem. In what follows we assume that $\tau \geq 5$. Our results show that

$$B_{\mathrm{odd}}(n, 2k-1) \gtrsim \frac{(1+\sqrt[2k-1]{3})}{(k-1)!} n^{k-1}$$

for $\tau = 2k-1$, $k = 3, 4, \ldots, 13$. Some discussion and numerical results for odd τ, $5 \leq \tau \leq 17$, are presented also.

2 Preliminaries

Let the integers $n \geq 3$, odd $\tau = 2k-1 \geq 3$, and odd $M \geq D(n,\tau)+1$ be fixed and let $C \in \mathbb{S}^{n-1}$ be a spherical τ-design of odd size $|C| = M = (\frac{2}{(k-1)!}+\gamma)n^{k-1}$, where $\gamma > 0$ is a constant.

For every fixed point $x \in C$ we consider $I(x) = \{\langle u, x\rangle : u \in C \setminus \{x\}\} = \{t_1(x), t_2(x), \ldots, t_{|C|-1}(x)\}$, where $-1 \leq t_1(x) \leq \cdots t_{|C|-1}(x) < 1$ (note that $I(x)$ may contain repeating numbers). We denote by $U_{\tau,i}(x)$ (respectively $L_{\tau,i}(x)$) any upper (resp. lower) bound on the inner product $t_i(x)$. When a bound does not depend on x we omit x in the notation.

It follows from [9, Section 4] (see also [2]) that for every fixed cardinality $M \geq D(n, 2k-1)$ there exist uniquely determined real numbers $-1 \leq \alpha_0 < \alpha_1 < \cdots < \alpha_{k-1} < 1$ and $\rho_0, \rho_1, \ldots, \rho_{k-1}$, $\rho_i > 0$ for $i = 0, 1, \ldots, k-1$, such that the equality

$$f_0 = \frac{f(1)}{M} + \sum_{i=0}^{k-1} \rho_i f(\alpha_i) \tag{1}$$

is true for every real polynomial $f(t)$ of degree at most $2k-1$. The numbers α_i, $i = 0, 1, \ldots, k-1$, are the roots of the equation $P_k(t)P_{k-1}(s) - P_k(s)P_{k-1}(t) = 0$, where $s = \alpha_{k-1}$, $P_i(t) = P_i^{(n-1)/2,(n-3)/2}(t)$ is a Jacobi polynomial [1].

We denote $g(t) = (t-\alpha_1)^2(t-\alpha_2)^2\cdots(t-\alpha_{k-1})^2 = \sum_{i=0}^{2k-2} g_i P_i^{(n)}(t)$. It follows by (3) that $g_0|C| - g(1) = \rho_0|C|g(\alpha_0)$.

Lemma 2.1. [2] *Let $C \subset \mathbb{S}^{n-1}$ be a τ-design with odd $\tau = 2k-1$. For any point $x \in C$ we have $t_1(x) \leq U_{\tau,1} = \alpha_0$ and $t_{|C|-1}(x) \geq L_{\tau,|C|-1} = \alpha_{k-1}$. If $|C|$ is odd then there exist a point $x \in C$ such that $t_2(x) \leq U_{\tau,2}(x) = \alpha_0$.*

Lemma 2.2. [3] *Let $C \subset \mathbb{S}^{n-1}$ be a τ-design with odd $\tau = 2k-1$ and odd cardinality $|C|$. Then there exist three distinct points $x, y, z \in C$ such that $t_1(x) = t_1(y)$ and $t_2(x) = t_1(z)$. Moreover, we have $t_{|C|-1}(z) \geq L_{\tau,|C|-1}(z) = \max\{\alpha_{k-1}, 2\alpha_0^2 - 1\}$.*

Theorem 2.3. [2] *If $C \subset \mathbb{S}^{n-1}$ is a τ-design with odd $\tau = 2k-1$ and odd $|C|$ then $\rho_0|C| \geq 2$.*

Every special triple $\{x, y, z\}$ from Lemma 2.2 is obviously and uniquely extended to a special quadruple $\{x, y, z, u\}$ by adding the point $u \in C$ which is defined by $t_2(z) = \langle z, u \rangle$. Our method is based on careful investigation of the special quadruples $\{x, y, z, u\} \in C$.

Definition 2.4. *A special quadruple $\{x, y, z, u\} \subset C$ is called "good" if $t_2(z) \leq \alpha_0$.*

Our main purpose is to obtain a bound $t_1(z) \leq U_{\tau,1}(z) < \alpha_0$. Such bounds start a procedure (of improving other bounds) which often reaches a contradiction with the existence of C. The inequality $U_{\tau,1}(z) < \alpha_0$ can be obtained in all cases: when a special quadruple which is not "good" exists, and when all special quadruples are "good".

We have (cf. [6]) $\alpha_0 \sim -\dfrac{1}{1+\gamma(k-1)!}$, $\alpha_i \sim 0$, for $i = 1, 2, \ldots, k-1$ and then $\rho_0|C| \sim (1+\gamma(k-1)!)^{2k-1}$. We also need the conditions $2 \leq \rho_0|C| < 3$ and $2\alpha_0^2 - 1 > \alpha_{k-1}$.

Let T_k be the set of the designs under consideration. We will always assume that the dimension n is large enough to have all bounds below valid.

3 Bounds for special triples and special quadruples

By Lemma 2.2 we have $t_{|C|-1}(z) \geq L_{\tau,|C|-1}(z) = 2\alpha_0^2 - 1$ in every special triple. We consecutively obtain the following bounds.

Lemma 3.1. [5] *If $\rho_0|C| < 3$ then $\alpha_0 < t_3(v)$ for every point $v \in C$.*

Lemma 3.2. *Let $C \subset T_k$ and $z \in C$ belongs to a special triple $\{x, y, z\}$ in C. Then we have $t_1(z) \geq L_{\tau,1}(z) \sim -\sqrt[2k-2]{\dfrac{1+\gamma(k-1)!}{2}}$, where $L_{\tau,1}(z)$ is the smallest root of the equation $2g(t) = \rho_0|C|g(\alpha_0)$.*

Lemma 3.3. *Let $C \subset T_k$ and $z \in C$ belongs to a special triple $\{x, y, z\}$ in C. If $\left(\dfrac{2}{(1+\gamma(k-1)!)^2} - 1\right)^{2k-2} < \dfrac{1+\gamma(k-1)!}{2}$ then*

$$t_2(z) \leq U_{\tau,2}(z) \sim \frac{\left(1 + \left(\frac{2}{(1+\gamma(k-1)!)^2} - 1\right)^{2k-1} - \frac{1+\gamma(k-1)!}{2}\right)\sqrt[2k-2]{\frac{1+\gamma(k-1)!}{2}}}{\left(\frac{2}{(1+\gamma(k-1)!)^2} - 1\right)^{2k-2} - \frac{1+\gamma(k-1)!}{2}}.$$

Lemma 3.4. [5] *a) If* $x, y_1, y_2 \in \mathbb{S}^{n-1}$ *are such that* $\langle x, y_1\rangle = a < 0$ *and* $\langle x, y_2\rangle = b < 0$*, then* $\langle y_1, y_2\rangle \geq ab - \sqrt{(1-a^2)(1-b^2)}$.

b) Let $C \subset T_k$ *and* $\{x, y, z\}$ *be a special triple in* C*. Then we have*

$$t_{|C|-1}(x) \geq L_{\tau,|C|-1}(x) = U_{\tau,1}(z)U_{\tau,2}(z) - \sqrt{(1-U^2_{\tau,1}(z))(1-U^2_{2k-1,2}(z))},$$

provided $U_{\tau,i}(z)$*,* $i = 1, 2$*, are negative upper bounds for* $t_i(z)$*,* $i = 1, 2$.

Lemma 3.5. *Let* $C \subset T_k$ *and* $\{x, y, z\}$ *be a special triple in* C*. Then we have*

$$\begin{aligned} t_3(z) \geq L_{\tau,3}(z) \quad \sim \quad & -[\frac{1+\gamma(k-1)!}{2} - \frac{1}{2}(\frac{1}{1+\gamma(k-1)!})^{2k-2} \\ & -\frac{1}{2}(\frac{2}{(1+\gamma(k-1)!)^2} - 1)^{2k-2}]^{\frac{1}{2k-2}}, \end{aligned}$$

where $L_{\tau,3}(z)$ *is the smallest root of the equation* $2g(t) = (\rho_0|C| - 1)g(\alpha_0) - g(2\alpha_0^2 - 1)$.

Let us have a special quadruple in C which is not "good ". Using the inequalities $t_1(z) \leq \alpha_0 < t_2(z)$ we obtain better upper bound $t_1(z) \leq U_{\tau,1}(z) < \alpha_0$.

Lemma 3.6. *Let* $C \subset T_k$ *and* $\{x, y, z, u\}$ *be a special quadruple in* C *which is not "good ". Then we have* $t_1(z) \leq U_{\tau,1}(z)$*, where* $f(t) = (t - L_{\tau,3}(z))g(t)$ *and* $U_{\tau,1}(z)$ *is the smallest root of the equation* $f(t) = (\rho_0|C| - 1)f(\alpha_0) - f(L_{\tau,|C|-1}(z))$.

Since $t_1(z) = t_2(x)$, Lemma 3.6. implies $t_2(x) \leq U_{\tau,2}(x) := U_{\tau,1}(z)$.

Lemma 3.7. *(x-check for existence of C) If*

$$L_x(g) := 1 + \gamma(k-1)! - 2(U_{\tau,2}(x))^{2k-2} - (L_{\tau,M-1}(x))^{2k-2} < 0,$$

then there exist no spherical τ*-designs* $C \subset T_k$*,* $|C| = M$*, with a special quadruple which is not "good ".*

If we have $L_x(g) \geq 0$ then we continue with a recursive procedure which replaces α_0 with $U_{\tau,1}(z)$ whenever possible. In all cases after several steps we obtain $L_x(g) < 0$ and nonexistence of the designs under consideration follows.

The case $t_2(z) \leq \alpha_0$ for every special quadruple $\{x, y, z, u\} \in C$ is considered in a similar way with the following stronger property.

Theorem 3.8. [4, 5] *Let* $C \subset T_k$ *in which all special quadruples are "good ". Then there exist a "good " special quadruple* $\{x, y, z, u\} \subset C$ *and a point* $v \in C\backslash\{x, y, z, u\}$ *such that* $\langle v, w\rangle \leq \alpha_0$ *for some* $w \in \{y, u\}$.

Corollary 3.9. [4, 5] *We have* $t_{|C|-2}(x) \geq 2\alpha_0^2 - 1$ *or* $t_{|C|-2}(z) \geq 2\alpha_0^2 - 1$ *for at least one "good " quadruple.*

The bounds from Corollary 3.9 in many cases turn out sufficiently good to prove nonexistence of the design C. In both cases we start with a good lower bound for $t_3(z)$ and this allows us to obtain the desired good upper bound on $t_1(z)$. Finally, certain analogs of Lemma 3.7 are applied.

4 Conclusions, comments and numerical results

It follows from the corresponding Lemmas and their proofs that all our bounds are monotonic in the right direction – the lower bounds are increasing and the upper bounds are decreasing. Also, the functionals $L_x(g)$ are decreasing. Therefore the nonexistence proof for any admissible γ_0 means nonexistence for every admissible $\gamma < \gamma_0$. The best γ we have achieved is $\gamma = \dfrac{\sqrt[2k-1]{3}-1}{(k-1)!}$ so for all odd $\tau \in \{5, 7, \ldots, 25\}$ we obtain nonexistence in all cases when $\rho_0|C| < 3$.

Theorem 4.1. *If $C \subset \mathbb{S}^{n-1}$ is a spherical τ-design, $\tau = 2k-1$, $k = 3, 4, \ldots, 13$, of odd cardinality and n is large enough, then $\rho_0|C| \geq 3$. In other words,*

$$B_{odd}(n, 2k-1) \gtrsim \frac{(1 + \sqrt[2k-1]{3})}{(k-1)!} n^{k-1}.$$

The next table gives asymptotic form of the Delsarte-Goethals-Seidel bounds [8], the previously best known bounds and the new bounds.

τ	Delsarte-Goethals-Seidel bounds [8]	Previously best known bounds	New bounds (with $\gamma = \frac{\sqrt[2k-1]{3}-1}{(k-1)!}$)
3	$2n$	$2.3925n$ [4]	
5	n^2	$1.09309n^2$ [3]	$1.12286n^2$
7	$\frac{n^3}{3} \approx 0.33333n^3$	$0.35314n^3$ [3]	$0.36165n^3$
9	$\frac{n^4}{12} \approx 0.08333n^4$	$0.08667n^4$ [7, 2, 6]	$0.08874n^4$
11	$\frac{n^5}{60} \approx 0.01666n^5$	$0.01721n^5$ [7, 2, 6]	$0.01754n^5$
13	$\frac{n^6}{360} \approx 0.0027777n^6$	$0.0028538n^6$ [7, 2, 6]	$0.0029003n^6$
15	$\frac{n^7}{2520} \approx 0.0003968n^7$	$0.0004062n^7$ [7, 2, 6]	$0.0004119n^7$
17	$\frac{n^8}{20160} \approx 0.00004960n^8$	$0.00005063n^8$ [7, 2, 6]	$0.00005126n^8$

Table 1. Asymptotic lower bounds for $B_{\mathrm{odd}}(n, \tau)$.

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