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# On the structure of binary orthogonal arrays with small covering radius

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**Abstract.** We obtain restrictions on the structure of binary orthogonal arrays of strength 5 under the assumption that their covering radius is close to the Fazekas-Levenshtein bound. We obtain lower and upper bounds on the number of the points of the array which are closest to a point of realization of the covering radius.

## 1 Introduction

Let $H(n,2)$ be the binary Hamming space of dimension $n$. An orthogonal array, or equivalently, a $\tau$-design $C$ in $H(n,2)$ is an $M \times n$ matrix of a code $C$ such that every $M \times \tau$ submatrix contains all ordered $\tau$-tuples of $H(\tau,2)$, each one exactly $\frac{|C|}{2^\tau}$ times as rows. It is well known that the strength is equal to the dual distance of $C$ minus one.

We consider $H(n,2)$ with the inner product $\langle x,y\rangle = 1 - \frac{2d(x,y)}{n}$, where $d(x,y)$ is the Hamming distance between $x$ and $y$. An equivalent definition of a $\tau$-designs (cf. [2, 3]) is convenient for the so called polynomial techniques.

**Definition 1.** A code $C \subset H(n,2)$ is a $\tau$-design in $H(n,2)$ if and only if every real polynomial $f(t)$ of degree at most $\tau$ and every point $y \in H(n,2)$ satisfy

$$\sum_{x\in C} f(\langle x,y\rangle) = f_0|C|, \tag{1}$$

where $f_0$ is the first coefficient in the expansion $f(t) = \sum_{i=1}^{n} f_i Q_i^{(n)}(t)$, $Q_i^{(n)}(t)$ are the normalized Krawtchouk polynomials, i.e.

$$Q_i^{(n)}(t) = \frac{1}{\binom{n}{i}} \sum_{j=0}^{i} (-1)^j \binom{d}{j}\binom{n-d}{i-j},\ i = 0,1,\ldots,n,$$

where $d = n(1-t)/2$ [1, 2, 3].

**Definition 2.** The number $\rho(C) = \max_{y \in H(n,2)} \min_{x \in C} d(x,y)$ is called covering radius of $C$.

We work with the covering radius in terms of the inner products as $t_c = 1 - \frac{2\rho(C)}{n} = \min_{y \in H(n,2} \max_{x \in C} \langle x,y \rangle$ which will be also called covering radius of $C$. Fazekas-Levenshtein [2, Theorem 2] obtain the following lower bound on $t_c$ (i.e. upper bound on $\rho(C)$): if $C$ is a $(2k-\varepsilon)$-design, then

$$t_c \geq t_{FL} = t_k^{0,1-\varepsilon}, \tag{2}$$

where $t_k^{0,1-\varepsilon}$ is the largest zero of certain polynomial.

Denote by $p_c(y)$ the number of the points $x \in C$ such that $t_c = \langle x,y \rangle = t_{|C|}(y)$. So $y$ is a point in $H(n,2)$ where the covering radius is attained – so called deep hole of $C$. In this note we consider designs with covering radius which is close to the bound $t_{FL}$ and obtain bounds on $p_c(y)$ for every possible $y$.

## 2 Bounds on $p_c(y)$

For every real number $a$ we denote by $[a]^{(n)}$ the minimum number $-1 + \frac{2\ell}{n}$, $\ell \in \mathbb{Z}$, which is greater than or equal to $a$ and, similarly, by $[a]_{(n)}$ the maximum number $-1 + \frac{2\ell}{n}$, $\ell \in \mathbb{Z}$, which is less than or equal to $a$. Therefore the Fazekas-Levenshtein bound (2) states $t_c \geq [t_{FL}]^{(n)}$.

For $y \in H(n,2)$ we define the (possibly) multiset

$$I(y) = \{\langle x,y \rangle : x \in C\} = \{t_1(y), t_2(y), ..., t_{|C|}(y)\},$$

where $-1 \leq t_1(y) \leq t_2(y) \leq \cdots \leq t_{|C|}(y) \leq 1$. It is clear that we can order the points of $C$ to achieve the ordering of $I(y)$ as required. In what remains, the point $y$ will be a point $y \in H(n,2)$, where the covering radius is realized, i.e. $t_{|C|}(y) = t_c$.

The next theorem gives a lower bound on $p_c(y)$ for designs whose covering radius is as close as possible to $t_{FL}$.

**Theorem 1.** *Let $C \subset H(n,2)$ be a $\tau$-design with covering radius $t_c = [t_{FL}]^{(n)}$. Let $f(t)$ be a real polynomial of degree at most $\tau$ such that $f(t) \leq 0$ for $t \in [-1, t_c - \frac{2}{n}]$ and $f(t)$ is increasing in $[t_c - \frac{2}{n}, t_c]$. Then*

$$p_c(y) \geq \frac{f_0|C|}{f(t_c)}$$

*for every admissible $y$.*

*Proof.* It follows by (1) and the conditions of the theorem that

$$f_0|C| = \sum_{i=1}^{|C|} f(t_i(y)) \leq p_c(y) f(t_c).$$

Since $f(t_c) > 0$, it follows that $p_c(y) \geq \frac{f_0|C|}{f([t_{FL}]^{(n)})}$.

If we relax the condition on $t_c$, then we can obtain only upper bounds on $p_c(y)$.

**Theorem 2.** *Let $C \subset H(n,2)$ be a $\tau$-design with covering radius $t_c \geq [t_{FL}]^{(n)}$. Let $f(t)$ be a real polynomial of degree at most $\tau$ such that $f(t) \geq 0$ for $t \in [-1,1]$ and $f(t)$ is increasing in $[[t_{FL}]^{(n)}, 1]$. Then*

$$p_c(y) \leq \frac{f_0|C|}{f(t_c)}$$

*for every admissible $y$.*

*Proof.* Similar to Theorem 1.

As usually in the polynomial techniques, the polynomials in Theorems 1 and 2 have some free parameters which must be optimized. For small degrees (strengths) this is a routine calculation which can be performed by Maple or Mathematika.

# 3 Some applications

To avoid trivial case we give examples with $\tau = 5$ (where trivial cases also occur, indeed).

We apply Theorem 1 with polynomials $f(t) = (t - [t_{FL}]^{(n)} + \frac{2}{n})(t^2 + at + b)^2$, where $a$ and $b$ will be determined in a way to satisfy the conditions for $f(t)$ and to maximize the function $F(a,b) = \frac{f_0|C|}{f(t_c)}$, $t_c = [t_{FL}]^{(n)}$. The maximum is obtained for

$$a_1 = \frac{4(n-1)(n-2k-2)(n-2k-1)(n-2k)}{A},$$

$$b_1 = -\frac{(6+8k+4k^2-7n-4kn+n^2)(2+4k^2-3n-4kn+n^2)}{A},$$

where $A = n(n^4 - 4n^3(2k+1) + n^2(24k^2+24k+5) - 2n(16k^3+24k^2+4k+1) + 8k(2k^3+4k^2+k-1)$. The explicit form of $F(a_1,b_1)$ is too long to be stated here.

We consider Theorem 2 for polynomials $f(t) = (t+1)(t^2+at+b)^2$, where $a$ and $b$ will be determined in a way to satisfy the conditions for $f(t)$ and to minimize the function $G(a,b)=\frac{f_0|C|}{f(t_c)}$, $t_c=1-\frac{2k}{n}$. Our calculations show that the minimum is obtained for

$$a_2=\frac{4k(n-2k)}{n(2+4k+4k^2-3n-4kn+n^2)},$$

$$b_2=-\frac{(n-2)(2+4k^2-3n-4kn+n^2)}{n^2(2+4k+4k^2-3n-4kn+n^2)}$$

and is equal to

$$G(a_2,b_2)=\frac{n(n-1)(n-2)|C|}{(n-k)B},$$

where $B=n^4-4n^3(2k+1)+n^2(24k^2+24k+7)-8n(4k^3+6k^2+2k+1)+4(4k^4+8k^3+4k^2+1)$.

It is worth to note that for length $n=9$, there is a coincidence $t_c=[t_{FL}]^{(n)}$. The upper and lower bounds by Theorems 1 and 2 also coincide and, moreover, give non-integral values. Therefore the bound $[t_{FL}]^{(n)}$ can not be attained. One possible value for $t_c$ remains and this gives the exact value of the covering radius $\rho(C)=1$.

In other cases, we obtain lower and upper bounds for $p_c$. Such bounds can be used for reducing the number of different cases in the following approach. We set $f(t)=1,t,\ldots,t^5$ in (1) and obtain a system of linear equations with unknowns – the numbers of the distance distribution of $C$ with respect to $y$. There are finitely many candidates for solutions of this system and their number is substantially reduced by using the restrictions on $p_c$. One preliminary step reduces the possible values of $p_{t_c-2/n}(y)=|\{x\in C:\langle x,y\rangle=t_c-\frac{2}{n}\}|$ by using the inequality

$$f_0|C|=\sum_{i=0}^{|C|}f(t_i(y))\geq p_{t_c-2/n}(y)f(t_c-\frac{2}{n})+p_c(y)f(t_c),$$

where $f(t)=(t+1)(t^2+at+b)^2$ is as in Theorem 2. This implies

$$p_{t_c-2/n}(y)\leq\frac{f_0|C|-p_c(y)f(t_c)}{f(t_c-2/n)}.$$

For example, for $n=10$ and $|C|=192$, under the assumption $t_c=[t_{FL}]^{(n)}=\frac{3}{5}$, we obtain $16\leq p_c\leq 21$ by the above calculations (applications of Theorems 1 and 2).

The corresponding systems for $p_c=16$, 17 and 21 do not have integer solutions and we conclude that $18\leq p_c\leq 20$. In these cases we obtain six solutions in total. In particular, we obtain no solutions with inner product $-1$, which means that $-y \notin C$ for any choice of $y$.

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