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On a class of functions in finite algebras

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Abstract. Given an n-ary $k-$valued function f, $gap(f)$ denotes the minimal number of essential variables in f which become fictive when identifying any two distinct essential variables in f. It is called the essential arity gap of f. We obtain an explicit determination of n-ary $k-$valued functions f whose essential arity gap is equal to m, $m \leq n \leq k$. Our methods yield new combinatorial results about the number of $k-$valued functions with given gap.

1 Introduction

Given a function f, the essential variables in f are defined as variables which occur in f and weigh with the values of that function. The number of essential variables is an important measure of complexity for discrete functions.

We proved a few results concerning simplifying of functions by identification of variables.

2 Preliminaries

Let k be a natural number with $k > 2$ and let $K = \{0, 1, \ldots, k-1\}$ be the set (ring) of remainders modulo k. An *n-ary k-valued function (operation) on K* is a mapping $f : K^n \to K$ for a natural number n, called *the arity* of f. The set of the all such functions is denoted by P_k^n.

Definition 2.1 *Let $X_n = \{x_1, \ldots, x_n\}$ be the set of n variables. A variable x_i is called essential in f, or f essentially depends on x_i, if there exist values $a_1, \ldots, a_n, b \in K$, such that*

$$f(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n) \neq f(a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_n).$$

The set of all essential variables in a function f is denoted by $Ess(f)$ and the number of its essential variables is denoted by $ess(f) = |Ess(f)|$.

Let x_i and x_j be two distinct essential variables in f. We say that the function g is obtained from $f \in P_k^n$ by *the identification of the variable x_i with x_j*, if

$$g = f(x_1, \ldots, x_{i-1}, x_j, x_{i+1}, \ldots, x_n) = f(x_i = x_j).$$

Briefly, when g is obtained from f, by identification of the variable x_i with x_j, we will write $g = f_{i\leftarrow j}$ and g is called *the identification minor of* f and $Min(f)$ denotes the set of all identification minors of f.

We shall allow formation of identification minors when x_i or x_j are not essential in f, also. Such minors of f are called *trivial* and they do not belong to $Min(f)$. So, if x_i does not occur in f, then we define $f_{i\leftarrow j} := f$.

Clearly, $ess(f_{i\leftarrow j}) \leq ess(f)$, because $x_i \notin Ess(f_{i\leftarrow j})$, even though it might be essential in f.

Definition 2.2 *Let $f \in P_k^n$ be an n-ary k-valued function. Then the essential arity gap (shortly arity gap or gap) of f is defined by*

$$gap(f) := ess(f) - \max_{g\in Min(f)} ess(g).$$

We let $G_{p,k}^m$ denote the set of all functions in P_k^n which essentially depend on m variables whose arity gap is p i.e. $G_{p,k}^m = \{f \in P_k^n \mid ess(f) = m \ \&\ gap(f) = p\}$, with $m \leq n$.

In [2] the Boolean functions whose arity gap is 2 are described. In [3] the class $G_{2,2}^n$ is investigated, also and several combinatorial results concerning the number of the functions in this class are obtained.

The case $2 \leq p \leq n$ and $n > k$ is fully described in [4] where it is proved that $gap(f) \leq 2$ and if $f \in G_{2,k}^n$ then f is a totally symmetric function.

So, we shall pay attention to the case $2 < k$ and $n \leq k$, solving a problem of M. Couceiro and E. Lehtonen, namely:
For each $1 \leq m \leq |A|$, determine explicitly the functions $f : A^n \to B$ whose arity gap is m ([1], page 6, Problem 1).

We shall assume that $A = B = K$. The most of the results obtained in this case might be easily generalized about finite defined and finite valued functions.

Let $m \in N$, $0 \leq m \leq k^n - 1$ be an integer. It is well known that for every $k, n \in N,\ k \geq 2$ there is an unique finite sequence $(\alpha_1, \ldots, \alpha_n) \in K^n$ such that

$$m = \alpha_1 k^{n-1} + \alpha_2 k^{n-2} + \ldots + \alpha_n.$$

This equation is known as the representation of m in $k-$ary positional numerical system. One briefly writes $m = \overline{\alpha_1\alpha_2\ldots\alpha_n}$.

Given a variable x and $\alpha \in K$, x^α is an important function defined by:

$$x^\alpha = \begin{cases} 1 & if \quad x = \alpha \\ 0 & if \quad x \neq \alpha. \end{cases}$$

In this paper we shall use *sums of conjunctions (SC)* for representation of functions in P_k^n. This is the most natural representation of the functions in finite algebras. It is based on so called operation tables of the functions.

Each function $f \in P_k^n$ can be uniquely represented in SC-form as follows

$$f = a_0.x_1^0 \dots x_n^0 \oplus \dots \oplus a_m.x_1^{\alpha_1} \dots x_n^{\alpha_n} \oplus \dots \oplus a_{k^n-1}.x_1^{k-1} \dots x_n^{k-1}$$

with $m = \overline{\alpha_1\alpha_2\dots\alpha_n}$, and $\alpha_i, a_m \in K$, where " $\oplus$ " and "." are the operations addition and multiplication modulo k in the ring K.

3 Essential arity gap of k-valued functions

First, we study the n-ary k-valued functions whose arity gap is n.

Given two natural numbers $k, n \geq 2$, Eq_k^n denotes the set of all strings over $K = \{0, 1, \dots, k-1\}$ with length n which have at least two equal letters i.e.

$$Eq_k^n := \{\alpha_1 \dots \alpha_n \in K^n \mid \alpha_i = \alpha_j, \text{ for some } i, j \leq n, \ i \neq j\}.$$

Theorem 3.1 *Let $f \in P_k^n$, be a function which depends essentially on all of its n variables and $2 < n \leq k$. Then $f \in G_{n,k}^n$ if and only if it can be represented as follows*

$$f = [\bigoplus_{\beta_1\dots\beta_n \notin Eq_k^n} a_r.x_1^{\beta_1} \dots x_n^{\beta_n}] \oplus a_0.[\bigoplus_{\alpha_1\dots\alpha_n \in Eq_k^n} x_1^{\alpha_1} \dots x_n^{\alpha_n}], \tag{1}$$

where $r = \overline{\beta_1 \dots \beta_n}$ and at least two among the coefficients $\{a_0\} \cup \{a_r \mid r = \overline{\beta_1 \dots \beta_n}, \ \& \ \beta_1 \dots \beta_n \notin Eq_k^n\}$, are distinct.

Corollary 3.1 *If $f \in G_{n,k}^n$ and $2 \leq n \leq k$, then $f(\alpha_1, \dots, \alpha_n) = f(0, \dots, 0)$ for all $\alpha_1 \dots \alpha_n \in Eq_k^n$.*

Corollary 3.2 *For each k, $k \geq 3$ the functions*

$$q_k(x_1, \dots, x_k) = \bigoplus_{\alpha_1\dots\alpha_k \in Eq_k^k} x_1^{\alpha_1} \dots x_k^{\alpha_k}$$

and

$$p_k(x_1, \dots, x_k) = \bigoplus_{\alpha_1\dots\alpha_k \notin Eq_k^k} a_m.x_1^{\alpha_1} \dots x_k^{\alpha_k},$$

where $m = \overline{\alpha_1 \dots \alpha_k}$ and at least two among the coefficients a_m are distinct, have the essential arity gap equal to k.

Theorem 3.2 *If $2 \leq n \leq k$ then*

$$|G_{n,k}^n| = k^{\left(\binom{k}{n}.n!+1\right)} - k.$$

The basic properties of the functions whose arity gap is p with $2 < p \leq n \leq k$, are described by the next theorems.

Theorem 3.3 *Let $f \in G^n_{p,k}$ with $2 < p \leq n \leq k$. Then*
(i) If $Ess(f_{u\leftarrow v}) \cap \{x_u, x_v, x_w\} = \emptyset$ and $ess(f_{u\leftarrow v}) = n - p$ then the following holds:

(i_1) $ess(f_{u\leftarrow w}) = ess(f_{v\leftarrow w}) = n - p$;

(i_2) $x_u \notin Ess(f_{v\leftarrow w})$ and $x_v \notin Ess(f_{u\leftarrow w})$;

(i_3) $x_j \in Ess(f_{v\leftarrow j})$ for all x_j, $x_j \in Ess(f_{u\leftarrow v})$;

(i_4) $x_i \notin Ess(f_{v\leftarrow i})$ for all i, $x_i \notin Ess(f_{u\leftarrow v})$.

(ii) If $x_v \in Ess(f_{u\leftarrow v})$, $Ess(f_{u\leftarrow v}) \cap \{x_u, x_w\} = \emptyset$ and $ess(f_{u\leftarrow v}) = n - p$ then the following holds:

(ii_1) $ess(f_{w\leftarrow v}) = ess(f_{w\leftarrow u}) = n - p$;

(ii_2) $x_v \notin Ess(f_{w\leftarrow u})$ and $x_u \notin Ess(f_{w\leftarrow v})$;

(ii_3) $x_v \in Ess(f_{w\leftarrow j}) \iff x_u \notin Ess(f_{w\leftarrow j})$ for all x_j, $x_j \notin \{x_u, x_v, x_w\}$;

(ii_4) $x_i \notin Ess(f_{w\leftarrow i})$ for all x_i, $x_i \notin Ess(f_{u\leftarrow v})$.

(ii_5) $x_j \in Ess(f_{w\leftarrow j})$ for all x_i, $x_i \in Ess(f_{u\leftarrow v})$.

Theorem 3.4 *Let $f \in G^n_{p,k}$ and $2 < p \leq n \leq k$. Then the following conditions hold:*

(i) There exist $u, v \in \{1, \ldots, n\}$ such that $f_{u\leftarrow v}$ depends essentially on $n - p$ variables and $x_v \in Ess(f_{u\leftarrow v})$;

(ii) There exist $u, v \in \{1, \ldots, n\}$ such that $f_{u\leftarrow v}$ depends essentially on $n - p$ variables and $x_v \notin Ess(f_{u\leftarrow v})$.

Remark 3.1 *The results from Theorem 3.3, and Theorem 3.4 might be summarized as follows:*

(i) Let $f \in G^n_{p,k}$. If $x_v \notin Ess(f_{u\leftarrow v})$ and $ess(f_{u\leftarrow v}) = n - p$, then $x_j \in Ess(f_{i\leftarrow j})$ and $x_i \in Ess(f_{j\leftarrow i})$ for all $x_j \in Ess(f_{u\leftarrow v})$ and $x_i \notin Ess(f_{u\leftarrow v})$, according to Theorem 3.3 (i).

(ii) Let $f \in G^n_{p,k}$. If $x_v \in Ess(f_{u\leftarrow v})$ and $ess(f_{u\leftarrow v}) = n - p$, then $x_i \notin Ess(f_{j\leftarrow i})$ for all $x_i, x_j \notin Ess(f_{u\leftarrow v})$, according to Theorem 3.3 (ii).

(iii) Thus, we have obtained a partition of the set $X_n = \{x_1, \ldots, x_n\}$ into the sets $V := Ess(f_{u\leftarrow v})$ and $W =: X_n \setminus V$, such that

$$(x_i, x_j) \in W^2 \Rightarrow (ess(f_{i\leftarrow j}) = n - p \ \&\ x_j \notin Ess(f_{i\leftarrow j}))$$

and

$$(x_i, x_j) \in W \times V \Rightarrow (ess(f_{i\leftarrow j}) = n - p \ \&\ x_j \in Ess(f_{i\leftarrow j})).$$

Theorem 3.5 *Let f be a k-valued function which depends essentially on the all of its n variables and $2 < p < n \leq k$. Then $f \in G^n_{p,k}$ if and only if there exist $n - p$ variables $y_{i_1}, \ldots, y_{i_{n-p}} \in X_n$ such that*

$$f(x_1, \ldots, x_n) = h(y_{i_1}, \ldots, y_{i_{n-p}}) \ \oplus \ g(x_1, \ldots, x_{n-p}, x_{n-p+1}, \ldots, x_n),$$

where h depends essentially on all of its $n - p$ variables and $g \in G^n_{n,k}$.

Theorem 3.6 *If $2 < p < n \leq k$, then*

$$|G^n_{p,k}| = [k^{\left(\binom{k}{n}.n!+1\right)} - k].\sum_{j=p}^{n}(-1)^{j-p}\binom{j}{p}\binom{n}{j}.k^{k^{n-j}}.$$

In the special class $G^3_{2,k}$ we have proved the following proposition.

Proposition 3.1 $|G^3_{2,k}| = 8.729.\binom{k}{3}.(k^k - k) = 5832.\binom{k}{3}.(k^k - k)$.

In the more general case of the class of functions $G^n_{2,k}$, $n > 3$ it might be proved the following combinatorial result:

Proposition 3.2 $|G^n_{2,k}| \geq [k^{\left(\binom{k}{n}.n!+1\right)} - k].\sum_{j=2}^{n}(-1)^j\binom{j}{2}\binom{n}{j}.k^{k^{n-j}}$.

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