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The uniqueness of the optimal (28,8,2,3) and (30,10,2,3) superimposed codes[1]

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Abstract. In this paper we prove that the optimal $(28, 8, 2, 3)$ and $(30, 10, 2, 3)$ superimposed codes are unique.

1 Introduction

Definition 1 *A binary $N \times T$ matrix $C = (c_{ij})$ is called an (N, T, w, r) superimposed code (SIC) if for any pair of subsets $W, R \subset \{1, ..., T\}$ such that $|W| = w$, $|R| = r$ and $W \bigcap R = \emptyset$, there exists a row $i \in \{1, 2, ..., N\}$ such that $c_{ij} = 1$ for all $j \in W$ and $c_{ij} = 0$ for all $j \in R$.*

The trivial code is a simple example for an (N, T, w, r) superimposed code. The length N of the trivial code is $\binom{T}{w}$ and its rows are all possible binary vectors of weight w.

Let $N(T, w, r)$ is the minimum length of an (N, T, w, r) superimposed code for given values of T, w and r. The code is called optimal when $N = N(T, w, r)$. The exact values of $N(T, 2, 3)$ are known for $T \leq 8$ and for $T = 10$ [2], [3].

T	5	6	7	8	9	10
$N(T, 2, 3)$	10	15	21	28	$28 - 30$	30

In this paper we prove that the optimal $(28, 8, 2, 3)$ and $(30, 10, 2, 3)$ superimposed codes are unique and $N(9, 2, 3) = 30$. The results have been obtained using the author's computer programs for the generation of $(N, T, 1, 2)$, $(N, T, 1, 3)$, $(N, T, 2, 2)$ and $(N, T, 2, 3)$ superimposed codes and the program *Q-extension* [1] for code equivalence testing.

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2 Preliminaries

Definition 2 *Two (N, T, w, r) superimposed codes are equivalent if one of them can be transformed into the other by a permutation of the rows and a permutation of the columns.*

Suppose C is an (N, T, w, r) superimposed code and x is a column of C. We denote by $wt(x)$ the Hamming weight of the column x.

Definition 3 *The residual code $Res(C, x = v)$ of C with respect to value v in column x is a code obtained by taking all the rows in which C has value v in column x and deleting the x^{th} entry in the selected rows.*

Lemma 4
(a) $Res(C, x = 0)$ is an $(N - wt(x), T - 1, w, r - 1)$ superimposed code;
(b) $Res(C, x = 1)$ is a $(wt(x), T - 1, w - 1, r)$ superimposed code.

Lemma 5 *Let C be an (N, T, w, r) superimposed code and x be a column of C. The matrix, obtaining of C by deleting of column x, is an $(N, T - 1, w, r)$ superimposed code.*

Lemma 6 *[2]*
$N(7, 1, 3) = 7$, $N(8, 1, 3) = 8$, $N(9, 1, 3) = 9$,
$N(7, 2, 2) = 14$, $N(8, 2, 2) = 14$, $N(9, 2, 2) = 18$, $N(7, 2, 3) = 21$.

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Lemma 7 *Each $(N, 7, 1, 3)$ superimposed code for $N = 7, 8, 9, 10$ and 11 contains at least 6 rows of weight 1.*

Proof We construct the $(N, 7, 1, 3)$ superimposed codes column by column, using the $(N, T, 1, 3)$ superimposed code generation program *Gen13SIC* and *Q-extension* for code equivalence testing. Using the method described in [4] we obtained the following number inequivalent $(N, 7, 1, 3)$ superimposed codes:

N	7	8	9	10	11
#	1	8	70	738	9484

Each of these codes contains at least 6 rows of weight 1. □

Using the author's programs *Gen22SIC* and *Gen23SIC* for generation of $(N, T, 2, 2)$ and $(N, T, 2, 3)$ superimposed codes and *Q-extension* for code equivalence testing we prove the following lemmas:

Lemma 8 *There exist exactly* 1 $(14, 7, 2, 2)$, 10 $(15, 7, 2, 2)$ *and* 152 $(16, 7, 2, 2)$ *inequivalent superimposed codes.*

Lemma 9
(a) Any $(21, 7, 2, 3)$ *SIC is equivalent to the trivial* $(21, 7, 2, 3)$ *SIC;*
(b) Any $(22, 7, 2, 3)$ *SIC contains as a submatrix the trivial* $(21, 7, 2, 3)$ *SIC.*

Theorem 10 *The trivial* $(28, 8, 2, 3)$ *superimposed code is the unique optimal* $(28, 8, 2, 3)$ *SIC.*

Proof It is proved in [3] that the trivial $(28, 8, 2, 3)$ SIC is optimal. Now we will prove that this code is the unique $(28, 8, 2, 3)$ SIC up to equivalence.

Let C be a $(28, 8, 2, 3)$ superimposed code and x be a column of C. The residual code $Res(C, x = 0)$ is an $(N_0, 7, 2, 2)$ SIC. According to Lemma 6 $N(7, 2, 2) = 14$, therefore $wt(x) \leq 14$. The residual code $Res(C, x = 1)$ is an $(N_1, 7, 1, 3)$ SIC. According to Lemma 6 $N(7, 1, 3) = 7$, therefore $wt(x) \geq 7$. We consider the following cases:

Case 1: There is a column y of C of weight 7.

The residual code $Res(C, y = 1)$ is $(7, 7, 1, 3)$ SIC. It is known [4] that the identity matrix of order 7 is the unique $(7, 7, 1, 3)$ SIC and all its rows are of weight 1. Therefore the residual code $Res(C, y = 0)$ is an $(21, 7, 2, 3)$ SIC. We extended the unique $(21, 7, 2, 3)$ SIC to a $(28, 8, 2, 3)$ SIC and obtained the trivial $(28, 8, 2, 3)$ SIC.

Case 2: All the columns of C are of weight between 8 and 14 and there is a column y of C of weight $v = 8, 9, 10$ or 11.

The residual code $Res(C, y = 1)$ is $(v, 7, 1, 3)$ SIC. According to Lemma 7 this residual code contains at least 6 rows of weight 1. If we delete the column y in C and 6 rows of weight 1 in the $Res(C, y = 1)$, we obtain the matrix with 22 rows and 7 columns which must be $(22, 7, 2, 3)$ SIC. According to Lemma 9 this code contains at most 49 ones. Therefore in C there is a column of weight at least 7 – a contradiction.

Case 3: All the columns of C are of weight between 12 and 14 and there is a column y of C of weight 14.

The residual code $Res(C, y = 0)$ is $(14, 7, 2, 2)$ SIC. According to Lemma 8 there is exactly 1 $(14, 7, 2, 2)$ SIC. So the matrix of the code C is of the form:

$$\left(\begin{array}{c|c} 0 & \\ \vdots & C_0 \\ 0 & \\ \hline 1 & \\ \vdots & C_1 \\ 1 & \end{array}\right)$$

where C_0 is the unique $(14, 7, 2, 2)$ SIC. Using an exhaustive computer search we tried to construct the matrix C_1. It turned out that the extension is impossible.

Case 4: All the columns of C are of weight between 12 and 13 and there is a column y of C of weight 13.

The residual code $Res(C, y = 0)$ is $(15, 7, 2, 2)$ SIC and the structure of the $(28, 8, 2, 3)$ code matrix is similar to that in Case 3. According to Lemma 8 there are 10 possibilities for the matrix C_0. For each of these cases we tried to construct C_1-part. It turned out, however, that there is no solution.

Case 5: All the columns of C are of weight 12. Let y be a column of C.

The residual code $Res(C, y = 0)$ is $(16, 7, 2, 2)$ SIC and the structure of the $(28, 8, 2, 3)$ code matrix is similar to that in Case 3. According to Lemma 8 there are 152 possibilities for the matrix C_0. For each of these cases we tried to construct C_1-part. It turned out that this is impossible. □

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Lemma 11 *Each $(N, 8, 1, 3)$ superimposed code for $N = 8, 9, 10$ and 11 contains at least 7 rows of weight 1.*

Proof We construct the $(N, 8, 1, 3)$ superimposed codes column by column, using the $(N, T, 1, 3)$ superimposed code generation program *Gen13SIC* and *Q-extension* for code equivalence testing. We obtained the following number inequivalent $(N, 8, 1, 3)$ superimposed codes:

N	8	9	10	11
#	1	9	95	1331

Each of these codes contains at least 7 rows of weight 1. □

Theorem 12 *There is no $(29, 9, 2, 3)$ superimposed code.*

Proof Let C be a $(29, 9, 2, 3)$ superimposed code and x be a column of C. The residual code $Res(C, x = 0)$ is an $(N_0, 8, 2, 2)$ SIC. According to Lemma 6 $N(8, 2, 2) = 14$, therefore $wt(x) \leq 15$. The residual code $Res(C, x = 1)$ is an $(N_1, 8, 1, 3)$ SIC. According to Lemma 6 $N(8, 1, 3) = 8$, therefore $wt(x) \geq 8$. We consider the following cases:

Case 1: All the columns of C are of weight between 8 and 15 and there is a column y of C of weight $v = 8, 9, 10$ or 11.

The residual code $Res(C, y = 1)$ is $(v, 8, 1, 3)$ SIC. According to Lemma 11 this residual code contains at least 7 rows of weight 1. If we delete the column y in C and 7 rows of weight 1 in the $Res(C, y = 1)$, we obtain the matrix with 22 rows and 8 columns which must be $(22, 8, 2, 3)$ SIC. This is a contradiction because $N(8, 2, 3) = 28$.

Case 2: All the columns of C are of weight between 12 and 15 and there is a column y of C of weight 12.

The residual code $Res(C, y = 0)$ is $(17, 8, 2, 2)$ SIC and the structure of the $(29, 9, 2, 3)$ code matrix is similar to that in Theorem 10. We classified up to equivalence all $(17, 8, 2, 2)$ superimposed codes and obtained that there are 4367 possibilities for the matrix C_0. For each of these cases we tried to construct C_1-part. It turned out that this is impossible.

Case 3: All the columns of C are of weight between 13 and 15 and there is a column y of C of weight 13.

The residual code $Res(C, y = 0)$ is $(16, 8, 2, 2)$ SIC and the structure of the $(29, 9, 2, 3)$ code matrix is similar to that in Theorem 10. We classified up to equivalence all $(16, 8, 2, 2)$ superimposed codes and obtained that there are 157 possibilities for the matrix C_0. For each of these cases we tried to construct C_1-part. It turned out, however, that there is no solution.

Case 4: All the columns of C are of weight between 14 and 15 and there is a column y of C of weight 14.

The residual code $Res(C, y = 0)$ is $(15, 8, 2, 2)$ SIC and the structure of the $(29, 9, 2, 3)$ code matrix is similar to that in Theorem 10. We classified up to equivalence all $(15, 8, 2, 2)$ superimposed codes and obtained that there are 10 possibilities for the matrix C_0. For each of these cases we tried to construct C_1-part. It turned out that this is impossible.

Case 5: All the columns of C are of weight 15. Suppose y is a column of C of weight 15.

The residual code $Res(C, y = 0)$ is $(14, 8, 2, 2)$ SIC and the structure of the $(29, 9, 2, 3)$ code matrix is similar to that in Theorem 10. There is unique $(14, 8, 2, 2)$ SIC. Using an exhaustive computer search we tried to construct the matrix C_1. It turned out that the extension is impossible. □

Theorem 13 $N(9, 2, 3) = 30$.

Proof It follows from Theorem 12 and $N(10, 2, 3) = 30$ [2]. □

5 The uniqueness of the optimal (30,10,2,3) superimposed code

It is known that the $3-(10, 4, 1)$ design is optimal $(30, 10, 2, 3)$ SIC [2]. In this section we will prove that:

Theorem 14 *The* $3-(10, 4, 1)$ *design is the unique* $(30, 10, 2, 3)$ *SIC.*

Proof Let C be a $(30, 10, 2, 3)$ superimposed code and x be a column of C. The residual code $Res(C, x = 0)$ is an $(N_0, 9, 2, 2)$ SIC. According to Lemma 6

$N(9,2,2) = 18$, therefore $wt(x) \leq 12$. The residual code $Res(C, x = 1)$ is an $(N_1, 9, 1, 3)$ SIC. According to Lemma 6 $N(9,1,3) = 9$, therefore $wt(x) \geq 9$. We consider the following cases:

Case 1: All the columns of C are of weight between 9 and 12 and there is a column y of C of weight $v = 9, 10$ or 11.

The residual code $Res(C, y = 1)$ is $(v, 9, 1, 3)$ SIC. We construct the $(v, 9, 1, 3)$ superimposed codes column by column, using the $(N, T, 1, 3)$ superimposed code generation program *Gen13SIC* and *Q-extension* for code equivalence testing. We obtained the following number inequivalent $(N, 9, 1, 3)$ superimposed codes:

N	9	10	11
#	1	10	125

Each of these codes contains at least 9 rows of weight 1. If we delete the column y in C and 9 rows of weight 1 in the $Res(C, y = 1)$, we obtain the matrix with 21 rows and 9 columns which must be $(21, 9, 2, 3)$ SIC. This is a contradiction because $N(9,2,3) = 30$.

Case 2: All the columns of C are of weight 12. Suppose y is a column of C of weight 12.

The residual code $Res(C, y = 0)$ is $(18, 9, 2, 2)$ SIC and the structure of the $(30, 10, 2, 3)$ code matrix is similar to that in Theorem 10. C_0 is the unique $(18, 9, 2, 2)$ SIC or its complementary code. Using an exhaustive computer search we constructed the matrix C_1 and we obtained that the $3 - (10, 4, 1)$ design is the unique $(30, 10, 2, 3)$ superimposed code. $\square$

References

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