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The nonexistence of some optimal arcs in PG(4, 4)

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Abstract. In this paper, we prove that there exist no arcs with parameters $(384, 97)$ and $(385, 97)$ in $\mathrm{PG}(4, 4)$. This implies the nonexistence of linear codes with parameters $[384, 5, 287]_4$ and $[385, 5, 288]_4$ and settles the problem of determining the exact value of $n_4(5, d)$ for $d = 267$ and 268, namely $n_4(5, 287) = 385$ and $n_4(5, 288) = 386$.

1 Introduction

In this paper, we consider the problem of determining the smallest possible length n of a linear $[n, k, d]_q$ codes of fixed dimension k and minimum distance d. The field $\mathbb{F}_q$ is tacitly assumed to be fixed, too. This particular value for n is denoted by $n_q(k, d)$. This problem is sometimes called the main coding theory problem.

The main coding theory problem for quaternary codes is solved for all $k \leq 4$ and for $k = 5$ for all but 120 values of d. In this paper, we tackle two of open cases for $q = 4, k = 5$, namely these with $d = 287$ and 288. We prove the nonexistence of linear codes with parameters $[384, 5, 287]_4$ and $[385, 5, 288]_4$ and thus settle the problem for these two values of d.

2 Preliminary results

For all the basic definitions and notations not explained, but used in this paper, we refer to [3] and [2].

Lemma 1. *Let $\mathfrak{K}$ be a* $(385, 97)$*-arc in* $\mathrm{PG}(4, 4)$*. Then we have*

$$\gamma_0(\mathfrak{K}) = 2, \gamma_1(\mathfrak{K}) = 7, \gamma_2(\mathfrak{K}) = 25, \gamma_3(\mathfrak{K}) = 97.$$

Lemma 2. *[5] The arcs described below are* $(118, 30)$*-arcs in* $\mathrm{PG}(3, 4)$*. Moreover every* $(118, 30)$*-arc in* $\mathrm{PG}(3, 4)$ *has one of the described types.*

(i) $\mathfrak{K} = \mathfrak{L} - \mathfrak{F}$, *where* $\mathfrak{L}$ *is a* $(128, 32)$*-arc and* $\mathfrak{F}$ *is a* $(10, 2)$*-minihyper (which in turn is a sum of two lines);*

(ii) Let L *be a line, let* π_i, $i = 0, \ldots, 4$, *be the planes through* L *and let* K_0, K_1, K_2 *be three lines in* π_0, π_1 *and* π_2, *respectively, meeting* L *in different points. Then*

$$\mathfrak{K}(x) = \begin{cases} 0 & \text{if } x \in \cup_{i=0}^{2} L \cap K_i, \\ 1 & \text{if } x \in \cup_{i=0}^{2} (K_i \setminus \pi_i) \cup (\cup_{i=3}^{4} (\pi_i \setminus \cup_{i=0}^{2} K_i), \\ 2 & \text{otherwise.} \end{cases}$$

(iii) Let L *be a line, let* π_i, $i = 0, \ldots, 4$, *be the planes through* L, *and let* x *be a point in* π_0. *Let further* $y_1, \ldots, y_6$ *be the points of an hyperoval in* π_1 *having* L *as an external line.*

$$\mathfrak{K}(x) = \begin{cases} 0 & \text{if } x \in L \cup \{x\}, \\ 1 & \text{if } x \in \cup_{i=1}^{4} (\pi_i \setminus (\cup_{j=1}^{6} \overline{xy_i} \cup L), \\ 2 & \text{otherwise.} \end{cases}$$

Lemma 3. *Let* $\mathfrak{K}$ *be a* $(97, 25)$*-arc in* $\mathrm{PG}(3, 4)$. *Then* $a_i = 0$ *for every* $i \neq 9, 13, 17, 21, 25$.

Proof. By Ward's divisibility theorem [6], the only possible intersection numbers for $\mathfrak{K}$ are: 1, 5, 9, 13, 15, 19, 21, and 25. 15- and 19-planes are ruled out by an easy counting (count the multiplicities of the points through a 4-, resp. 5-line). If there is an 1- or 5-plane, then by increasing the multiplicities of all points on this plane by 1, we get a $(118, 30)$-arc with a 22-plane or 26-plane without 0-points. A $(118, 30)$-arc with a plane with no 0-points has to be of type (ii). But the planes without 0-points of such arcs have multiplicity 30, a contradiction. Hence 1-planes and 5-planes are impossible. □

Lemma 4. *Let* $\mathfrak{K}$ *be a* $(385, 97)$*-arc in* $\mathrm{PG}(4, 4)$. *Then* $a_i = 0$ *for all* $i \neq 65, 81, 97$.

Proof. The usual counting argument gives that the possible multiplicities of hyperplanes (solids) are 1, 17, 33, 49, 65, 81, 97. Hyperplanes of multiplicity 1 and 17 are ruled out by Lemma 3. The restriction of $\mathfrak{K}$ to hyperplanes of multiplicity 33 has to be a $(33, 9)$ arc in $\mathrm{PG}(3, 4)$, which does not exist.

Assume there exists a 49-solid. The restriction of $\mathfrak{K}$ to such a solid is a $(49, 13)$-arc in $\mathrm{PG}(3, 4)$, which is known to have planes of multiplicities 1, 9, and 13 only. By Lemma 3, we get that through an 1-plane in a 49-solid there pass hyperplanes with at most 81 points. Hence

$$|\mathfrak{K}| \leq 1 + 48 + 4 \cdot 80 = 369 < 385,$$

a contradiction. □

If (a_i) denotes the spectrum of a $(385, 97)$-arc $\mathfrak{K}$ and λ_2 denotes the number of its double points then we have

$$\begin{array}{rcrcrcl} a_{97} & + & a_{81} & + & a_{65} & = & 341 \\ 97a_{97} & + & 81a_{81} & + & 65a_{65} & = & 85 \cdot 385 \\ \binom{97}{2}a_{97} & + & \binom{81}{2}a_{81} & + & \binom{65}{2}a_{65} & = & \binom{341}{2} \cdot 21 + 64\lambda_2 \end{array},$$

whence

$$15a_{81} + 62a_{65} = -198 + 8\lambda_2.$$

This gives the following possible spectra for a $(385, 97)$-arc in $\mathrm{PG}(4, 4)$:

a_{65}	a_{81}	a_{97}	λ_2
11	0	330	110
10	2	329	106
9	4	328	102
8	6	327	98
7	8	326	94
6	10	325	90
5	12	324	86
4	14	323	82
3	16	322	78
2	18	321	74
1	20	320	70
0	22	319	66

Lemma 5. *Let $\mathfrak{K}$ be a $(385, 97)$-arc in $\mathrm{PG}(4, 4)$. Then the possible multiplicities for planes in $\mathrm{PG}(4, 4)$ are 9, 13, 17, 21, or 25.*

Proof. If there is a plane π of multiplicity not equal to 9, 13, 17, 21, or 25, it has to be contained in 81- or 65-solids. But then, we have just two possibilities:

- π is of size 5 and the five solids through π are of multiplicity 81, and
- π is of size 1, and four of the solids through π are of multiplicity 81, and one is of multiplicity 65.

Now increasing the multiplicities of the points on π by 1 , we get a $(406, 102)$-arc. But such an arc does not exist (cf. [4]). □

Now we have the following possibilities for hyperplanes H_i, $i = 0, \ldots, 4$, through a fixed plane π:

	$\mathfrak{K}(\pi)$	$(\mathfrak{K}(\pi_0),\dots,\mathfrak{K}(\pi_4))$
(a)	25	$(97,97,97,97,97)$
(b)	21	$(97,97,97,97,81)$
(c)	17	$(97,97,97,97,65)$
(d)	17	$(97,97,97,81,81)$
(e)	13	$(97,97,97,81,65)$
(f)	13	$(97,97,81,81,81)$
(g)	9	$(97,97,97,65,65)$
(h)	9	$(97,97,81,81,65)$
(i)	9	$(97,81,81,81,81)$

Let us note that by Lemma 2, a 9-plane in a $(97,25)$-arc in $\mathrm{PG}(3,4)$ has two or three external lines.

Lemma 6. *Cases (g) and (h) are impossible.*

Proof. Consider a projection from a suitably chosen 0-line L in the 9-plane. By Lemma 2 it follows that:

• if the 9-plane contains two 0-lines the projection from either of them is of type $(25,25,25,13,9)$,

• if the plane contains three 0-lines the projection from two of them is of type $(25,25,25,13,9)$ and from one of them is $(25,21,21,21,9)$.

The image of the 65-solid is of type $(17,17,13,9,9)$ or $(17,13,13,13,9)$.

Case (g). Choose the line of projection in such way that at least two of the images of 97-planes are of type $(25,25,25,13,9)$. The line through the two points of multiplicity 13 has to be of type $(13,13,x,y,z)$, where $x \le 25$, $y,z \le 17$; $x = 25$ is impossible since a hyperplane with a 27-plane must be of multiplicity 97, but $13+13+25+17+17 < 97$. Further $x = 21$ is impossible since two 81-solids cannot meet in a 21-plane. Hence the third 97-line is of type $(25,25,25,13,9)$. But then $x \le 13$, $y,z \le 9$ and the line through the two 13-points is of multiplicity at most $13+13+13+9+9 < 65$, a contradiction.

Case (h). Again, the images of the 97-solids can be chosen to be of type $(25,25,25,13,9)$ and consider the line through two 13-points. It is of type $(13,13,x,y,z)$, $x,y \le 21$, $z \le 17$. As in case (g), this line is a 65-line and $x,y \le 13$ (since a 81- and a 65-solid meet in at most a 13-plane) and $z \le 9$ (since two 65-solids meet in at most a 9-plane). Now $13+13+13+9+9 < 65$, a contradiction. □

Corollary 7. *Let* $\mathfrak{K}$ *be a* $(385,97)$*-arc in* $\mathrm{PG}(4,4)$ *and the only possible spectra for* $\mathfrak{K}$ *are*

$$a_{65} = 1, a_{81} = 20, a_{97} = 320, \lambda_2 = 70,$$

$$a_{65} = 0, a_{81} = 22, a_{97} = 319, \lambda_2 = 66.$$

Corollary 8. *Let $\mathfrak{K}$ be a* $(385, 97)$*-arc in* $\mathrm{PG}(4, 4)$*. The restriction of $\mathfrak{K}$ to a 65-solid is a* $(65, 17)$*-arc in* $\mathrm{PG}(3, 4)$ *with spectrum* $a_{13} = 20, a_{17} = 65, \lambda_2 = 0$.

Proof. By Lemma 6, a 65-solid in $\mathfrak{K}$ cannot have a 9-plane. □

Lemma 9. *Case (i) is impossible.*

Proof. Consider a projection from such a 0-line in the 9-plane that the image of the 97-solid is of type $(25, 25, 25, 13, 9)$. Consider the line through the 13-point and a 21-point on one of the remaining four 81-lines. It is of type $(13, 21, x, y, z)$, $x, y, z \leq 21$, and must have multiplicity 97. Hence its type is $(13, 21, 21, 21, 21)$.

Consider the solid corresponding to this plane. The restriction of $\mathfrak{K}$ to this solid can be extended to a $(102, 26)$-arc by taking the points on the line of projection. This $(102, 26)$-arc in $\mathrm{PG}(3, 4)$ has an 18-plane, which is impossible since every $(102, 26)$-arc is the sum of the points of $\mathrm{PG}(3, 4)$ and a cap and has intersection numbers 22 and 26 (cf. [4]). □

3 The main theorem

In this section we prove our main nonexistence result.

Theorem 10. *There exists no* $(385, 97)$*-arc in* $\mathrm{PG}(4, 4)$.

Proof. Let $\mathfrak{K}$ be a $(385, 97)$ arc in $\mathrm{PG}(4, 4)$. Define a new arc $\mathfrak{K}^*$ by taking the solids of multiplicity w as points of multiplicity $(97 - w)/16$ in the dual space. It is an easy check that $\mathfrak{K}^*$ is a $(22, \{2, 6, 10\})$-arc in $\mathrm{PG}(4, 4)$, i.e. an arc of size 22 and intersection numbers 2, 6, and 10. Moreover, by Lemmas 6 and 9, a line has multiplicity at most 3.

The first step is to rule out the possibility of 3-lines in $\mathfrak{K}^*$. Note that this rules out the possibility of 2-points, and, consequently, of 65-solids in $\mathfrak{K}$. The projection form a 3-line is a plane $(19, \{3, 7\})$-arc. These arcs are easily classified. They are one of the following:

(1) the sum of three concurrent lines plus a point of multiplicity 4 off the lines;

(2) the sum of three non-concurrent lines plus a point of multiplicity 4 outside these lines;

(3) two 3-points and seven 1-points forming the sides of a triangle without the vertices (the three points are on the same side) plus three 2-points on the line connecting the opposing vertex to the side with the 3-points and the 1-point on the same side;

(4) two copies of a Baer subplane plus a tangent to it.

A straightforward computer search shows that none of them can be extended to a $(22, \{2, 6, 10\})$.

Now $\mathfrak{K}$ is an arc with two intersection numbers 81 and 97, and therefore $\mathfrak{K}^*$ is a $(22, \{6, 10\})$-arc in $\mathrm{PG}(4, 4)$ such that no three points are collinear. Consider a 6-solid and a 0-plane in it (such a plane necessarily exists). The solids through this plane have at least 6 points each, whence $|\mathfrak{K}| \geq 5.6 = 30$, a contradiction. □

Corollary 11. *There exists no* $(384, 97)$*-arc in* $\mathrm{PG}(4, 4)$.

Proof. Assume that $\mathfrak{K}$ is a $(384, 97)$-arc in $\mathrm{PG}(4, 4)$. It is easily checked that multiplicities of solids are congruent to n or $n+1$ (i.e. congruent to 0 or 1) modulo 4. Hence by the extension Theorem of Hill and Lizak [1], There exists a $(385, 97)$ arc, a contradiction to Theorem 10. □

Corollary 12. *There exist no linear codes with parameters* $[384, 5, 287]_4$ *and* $[385, 5, 288]_4$. *Consequently* $n_4(5, 287) = 385$ *and* $n_4(5, 288) = 386$.

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