The nonexistence of some optimal arcs in \( \text{PG}(4, 4) \)

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**Abstract.** In this paper, we prove that there exist no arcs with parameters \((384, 97)\) and \((385, 97)\) in \( \text{PG}(4, 4) \). This implies the nonexistence of linear codes with parameters \([384, 5, 287]_4\) and \([385, 5, 288]_4\) and settles the problem of determining the exact value of \(n_4(5, d)\) for \(d = 267\) and 268, namely \(n_4(5, 287) = 385\) and \(n_4(5, 288) = 386\).

1 Introduction

In this paper, we consider the problem of determining the smallest possible length \(n\) of a linear \([n, k, d]_q\) codes of fixed dimension \(k\) and minimum distance \(d\). The field \(\mathbb{F}_q\) is tacitly assumed to be fixed, too. This particular value for \(n\) is denoted by \(n_q(k, d)\). This problem is sometimes called the main coding theory problem.

The main coding theory problem for quaternary codes is solved for all \(k \leq 4\) and for \(k = 5\) for all but 120 values of \(d\). In this paper, we tackle two of open cases for \(q = 4, k = 5\), namely these with \(d = 287\) and 288. We prove the nonexistence of linear codes with parameters \([384, 5, 287]_4\) and \([385, 5, 288]_4\) and thus settle the problem for these two values of \(d\).

2 Preliminary results

For all the basic definitions and notations not explained, but used in this paper, we refer to \([3]\) and \([2]\).

**Lemma 1.** Let \(\mathcal{A}\) be a \((385, 97)\)-arc in \(\text{PG}(4, 4)\). Then we have  
\[
\gamma_0(\mathcal{A}) = 2, \gamma_1(\mathcal{A}) = 7, \gamma_2(\mathcal{A}) = 25, \gamma_3(\mathcal{A}) = 97.
\]

**Lemma 2.** \([5]\) The arcs described below are \((118, 30)\)-arcs in \(\text{PG}(3, 4)\). Moreover every \((118, 30)\)-arc in \(\text{PG}(3, 4)\) has one of the described types.
(i) \( \mathcal{R} = \mathcal{L} - \mathcal{F} \), where \( \mathcal{L} \) is a \((128, 32)\)-arc and \( \mathcal{F} \) is a \((10, 2)\)-minihyper (which in turn is a sum of two lines);

(ii) Let \( L \) be a line, let \( \pi_i, i = 0, \ldots, 4 \), be the planes through \( L \) and let \( K_0, K_1, K_2 \) be three lines in \( \pi_0, \pi_1 \) and \( \pi_2 \), respectively, meeting \( L \) in different points. Then

\[
\mathcal{R}(x) = \begin{cases} 
0 & \text{if } x \in \bigcup_{i=0}^2 L \cap K_i, \\
1 & \text{if } x \in \bigcup_{i=0}^2 (K_i \setminus \pi_i) \cup \bigcup_{i=3}^4 (\pi_i \setminus \bigcup_{i=0}^2 K_i), \\
2 & \text{otherwise}.
\end{cases}
\]

(iii) Let \( L \) be a line, let \( \pi_i, i = 0, \ldots, 4 \), be the planes through \( L \), and let \( x \) be a point in \( \pi_0 \). Let further \( y_1, \ldots, y_6 \) be the points of an hyperoval in \( \pi_1 \) having \( L \) as an external line.

\[
\mathcal{R}(x) = \begin{cases} 
0 & \text{if } x \in L \cup \{x\}, \\
1 & \text{if } x \in \bigcup_{i=1}^4 (\pi_i \setminus \bigcup_{j=1}^6 \pi_i \cup L), \\
2 & \text{otherwise}.
\end{cases}
\]

Lemma 3. Let \( \mathcal{R} \) be a \((97, 25)\)-arc in \( \text{PG}(3, 4) \). Then \( a_i = 0 \) for every \( i \neq 9, 13, 17, 21, 25 \).

Proof. By Ward’s divisibility theorem \([6]\), the only possible intersection numbers for \( \mathcal{R} \) are: 1, 5, 9, 13, 15, 19, 21, and 25. 15- and 19-planes are ruled out by an easy counting (count the multiplicities of the points through a 4-, resp. 5-line). If there is an 1- or 5-plane, then by increasing the multiplicities of all points on this plane by 1, we get a \((118, 30)\)-arc with a 22-plane or 26-plane without 0-points. A \((118, 30)\)-arc with a plane with no 0-points has to be of type (ii). But the planes without 0-points of such arcs have multiplicity 30, a contradiction. Hence 1-planes and 5-planes are impossible.

Lemma 4. Let \( \mathcal{R} \) be a \((385, 97)\)-arc in \( \text{PG}(4, 4) \). Then \( a_i = 0 \) for all \( i \neq 65, 81, 97 \).

Proof. The usual counting argument gives that the possible multiplicities of hyperplanes (solids) are 1, 17, 33, 49, 65, 81, 97. Hyperplanes of multiplicity 1 and 17 are ruled out by Lemma 3. The restriction of \( \mathcal{R} \) to hyperplanes of multiplicity 33 has to be a \((33, 9)\) arc in \( \text{PG}(3, 4) \), which does not exist.

Assume there exists a 49-solid. The restriction of \( \mathcal{R} \) to such a solid is a \((49, 13)\)-arc in \( \text{PG}(3, 4) \), which is known to have planes of multiplicities 1, 9, and 13 only. By Lemma 3, we get that through an 1-plane in a 49-solid there pass hyperplanes with at most 81 points. Hence

\[
|\mathcal{R}| \leq 1 + 48 + 4 \cdot 80 = 369 < 385,
\]
a contradiction.

If \((a_i)\) denotes the spectrum of a \((385, 97)\)-arc \(K\) and \(\lambda_2\) denotes the number of its double points then we have

\[
\begin{align*}
a_{97} + a_{81} + a_{65} &= 341 \\
97a_{97} + 81a_{81} + 65a_{65} &= 85 \cdot 385 \\
\binom{97}{2}a_{97} + \binom{81}{2}a_{81} + \binom{65}{2}a_{65} &= \binom{341}{2} \cdot 21 + 64\lambda_2
\end{align*}
\]

whence

\[
15a_{81} + 62a_{65} = -198 + 8\lambda_2.
\]

This gives the following possible spectra for a \((385, 97)\)-arc in \(\text{PG}(4, 4)\):

<table>
<thead>
<tr>
<th>(a_{65})</th>
<th>(a_{81})</th>
<th>(a_{97})</th>
<th>(\lambda_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>0</td>
<td>330</td>
<td>110</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>329</td>
<td>106</td>
</tr>
<tr>
<td>9</td>
<td>4</td>
<td>328</td>
<td>102</td>
</tr>
<tr>
<td>8</td>
<td>6</td>
<td>327</td>
<td>98</td>
</tr>
<tr>
<td>7</td>
<td>8</td>
<td>326</td>
<td>94</td>
</tr>
<tr>
<td>6</td>
<td>10</td>
<td>325</td>
<td>90</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>324</td>
<td>86</td>
</tr>
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<td>4</td>
<td>14</td>
<td>323</td>
<td>82</td>
</tr>
<tr>
<td>3</td>
<td>16</td>
<td>322</td>
<td>78</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
<td>321</td>
<td>74</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
<td>320</td>
<td>70</td>
</tr>
<tr>
<td>0</td>
<td>22</td>
<td>319</td>
<td>66</td>
</tr>
</tbody>
</table>

**Lemma 5.** Let \(K\) be a \((385, 97)\)-arc in \(\text{PG}(4, 4)\). Then the possible multiplicities for planes in \(\text{PG}(4, 4)\) are 9, 13, 17, 21, or 25.

**Proof.** If there is a plane \(\pi\) of multiplicity not equal to 9, 13, 17, 21, or 25, it has to be contained in 81- or 65-solids. But then, we have just two possibilities:

- \(\pi\) is of size 5 and the five solids through \(\pi\) are of multiplicity 81, and
- \(\pi\) is of size 1, and four of the solids through \(\pi\) are of multiplicity 81, and one is of multiplicity 65.

Now increasing the multiplicities of the points on \(\pi\) by 1, we get a \((406, 102)\)-arc. But such an arc does not exist (cf. [4]).

Now we have the following possibilities for hyperplanes \(H_i, i = 0, \ldots, 4\), through a fixed plane \(\pi\):
Let us note that by Lemma 2, a 9-plane in a (97, 25)-arc in PG(3, 4) has two or three external lines.

**Lemma 6.** Cases (g) and (h) are impossible.

**Proof.** Consider a projection from a suitably chosen 0-line \( L \) in the 9-plane. By Lemma 2 it follows that:

- if the 9-plane contains two 0-lines the projection from either of them is of type \((25, 25, 25, 13, 9)\),
- if the plane contains three 0-lines the projection from two of them is of type \((25, 25, 25, 13, 9)\) and from one of them is \((25, 21, 21, 21, 9)\).

The image of the 65-solid is of type \((17, 17, 13, 9, 9)\) or \((17, 13, 13, 13, 9)\).

**Case (g).** Choose the line of projection in such way that at least two of the images of 97-planes are of type \((25, 25, 25, 13, 9)\). The line through the two points of multiplicity 13 has to be of type \((13, x, y, z)\), where \(x \leq 25\), \(y, z \leq 17\); \(x = 25\) is impossible since a hyperplane with a 27-plane must be of multiplicity 97, but \(13 + 13 + 25 + 17 + 17 < 97\). Further \(x = 21\) is impossible since two 81-solids cannot meet in a 21-plane. Hence the third 97-line is of type \((25, 25, 25, 13, 9)\). But then \(x \leq 13\), \(y, z \leq 9\) and the line through the two 13-points is of multiplicity at most \(13 + 13 + 13 + 9 + 9 < 65\), a contradiction.

**Case (h).** Again, the images of the 97-solids can be chosen to be of type \((25, 25, 25, 13, 9)\) and consider the line through two 13-points. It is of type \((13, 13, x, y, z)\), \(x, y \leq 21\), \(z \leq 17\). As in case (g), this line is a 65-line and \(x, y \leq 13\) (since a 81- and a 65-solid meet in at most a 13-plane) and \(z \leq 9\) (since two 65-solids meet in at most a 9-plane). Now \(13 + 13 + 13 + 9 + 9 < 65\), a contradiction. \(\square\)

**Corollary 7.** Let \( \mathcal{R} \) be a \((385, 97)\)-arc in PG(4, 4) and the only possible spectra for \( \mathcal{R} \) are

\[
\begin{array}{c|cc}
\mathcal{R}(\pi) & (\mathcal{R}(\pi_0), \ldots, \mathcal{R}(\pi_4)) \\
\hline
(a) & 25 & (97, 97, 97, 97, 97) \\
(b) & 21 & (97, 97, 97, 97, 81) \\
(c) & 17 & (97, 97, 97, 97, 65) \\
(d) & 17 & (97, 97, 97, 81, 81) \\
(e) & 13 & (97, 97, 97, 81, 65) \\
(f) & 13 & (97, 97, 81, 81, 81) \\
(g) & 9 & (97, 97, 97, 65, 65) \\
(h) & 9 & (97, 97, 81, 81, 65) \\
(i) & 9 & (97, 81, 81, 81, 81)
\end{array}
\]

\[a_{65} = 1, a_{81} = 20, a_{97} = 320, \lambda_2 = 70, \]
\[a_{65} = 0, a_{81} = 22, a_{97} = 319, \lambda_2 = 66.\]
Corollary 8. Let $\mathcal{K}$ be a $(385, 97)$-arc in $\text{PG}(4, 4)$. The restriction of $\mathcal{K}$ to a 65-solid is a $(65, 17)$-arc in $\text{PG}(3, 4)$ with spectrum $a_{13} = 20, a_{17} = 65, \lambda_2 = 0$.

Proof. By Lemma 6, a 65-solid in $\mathcal{K}$ cannot have a 9-plane. \hfill \Box

Lemma 9. Case (i) is impossible.

Proof. Consider a projection from such a 0-line in the 9-plane that the image of the 97-solid is of type $(25, 25, 25, 13, 9)$. Consider the line through the 13-point and a 21-point on one of the remaining four 81-lines. It is of type $(13, 21, x, y, z)$, $x, y, z \leq 21$, and must have multiplicity 97. Hence its type is $(13, 21, 21, 21, 21)$.

Consider the solid corresponding to this plane. The restriction of $\mathcal{K}$ to this solid can be extended to a $(102, 26)$-arc by taking the points on the line of projection. This $(102, 26)$-arc in $\text{PG}(3, 4)$ has an 18-plane, which is impossible since every $(102, 26)$-arc is the sum of the points of $\text{PG}(3, 4)$ and a cap and has intersection numbers 22 and 26 (cf. [4]). \hfill \Box

3 The main theorem

In this section we prove our main nonexistence result.

Theorem 10. There exists no $(385, 97)$-arc in $\text{PG}(4, 4)$.

Proof. Let $\mathcal{K}$ be a $(385, 97)$ arc in $\text{PG}(4, 4)$. Define a new arc $\mathcal{K}^*$ by taking the solids of multiplicity $w$ as points of multiplicity $(97 - w)/16$ in the dual space. It is an easy check that $\mathcal{K}^*$ is a $(22, \{2, 6, 10\})$-arc in $\text{PG}(4, 4)$, i.e. an arc of size 22 and intersection numbers 2, 6, and 10. Moreover, by Lemmas 6 and 9, a line has multiplicity at most 3.

The first step is to rule out the possibility of 3-lines in $\mathcal{K}^*$. Note that this rules out the possibility of 2-points, and, consequently, of 65-solids in $\mathcal{K}$. The projection form a 3-line is a plane $(19, \{3, 7\})$-arc. These arcs are easily classified. They are one of the following:

1. the sum of three concurrent lines plus a point of multiplicity 4 off the lines;

2. the sum of three non-concurrent lines plus a point of multiplicity 4 outside these lines;

3. two 3-points and seven 1-points forming the sides of a triangle without the vertices (the three points are on the same side) plus three 2-points on the line connecting the opposing vertex to the side with the 3-points and the 1-point on the same side;
(4) two copies of a Baer subplane plus a tangent to it.

A straightforward computer search shows that none of them can be extended to a \((22, \{2, 6, 10\})\).

Now \(\mathcal{A}\) is an arc with two intersection numbers 81 and 97, and therefore \(\mathcal{A}'\) is a \((22, \{6, 10\})\)-arc in \(\text{PG}(4, 4)\) such that no three points are collinear. Consider a 6-solid and a 0-plane in it (such a plane necessarily exists). The solids through this plane have at least 6 points each, whence \(|\mathcal{A}'| \geq 5 \cdot 6 = 30\), a contradiction.

**Corollary 11.** There exists no \((384, 97)\)-arc in \(\text{PG}(4, 4)\).

**Proof.** Assume that \(\mathcal{A}\) is a \((384, 97)\)-arc in \(\text{PG}(4, 4)\). It is easily checked that multiplicities of solids are congruent to \(n\) or \(n+1\) (i.e. congruent to 0 or 1) modulo 4. Hence by the extension Theorem of Hill and Lizak [1], there exists a \((385, 97)\) arc, a contradiction to Theorem 10.

**Corollary 12.** There exist no linear codes with parameters \([384, 5, 287]_4\) and \([385, 5, 288]_4\). Consequently \(n_4(5, 287) = 385\) and \(n_4(5, 288) = 386\).

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**References**


