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Optimal arcs in Hjelmslev spaces of higher dimension

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Abstract. In this paper, we present various results on arcs in projective three-dimensional Hjelmslev spaces over finite chain rings of nilpotency index 2. A table is given containing exact values and bounds for projective arcs in the geometries over the two chain rings with four elements.

1 Introduction

From the point of view of coding theory, arcs in higher dimensional Hjelmslev spaces are of particular interest. However, the ongoing research is focused mainly on plane projective arcs. In this paper, we present some constructions and upper bounds on arcs in the 3-dimensional Hjelmslev spaces over the finite chain rings of nilpotency index 2. As a by-product, we obtain also new results on non-projective plane arcs. In order to save space, we have not introduced the basic facts on projective Hjelmslev geometries over finite chain rings. A self-contained introduction to coordinate Hjelmslev geometries is given e.g. in [2].

2 A recursive upper bound

Denote by $m_n(R_R^t)$ the largest size of a (k,n)-arc in $\mathrm{PHG}(R_R^t)$.

Theorem 1. *Let $\mathfrak{K}$ be a (k,n)-(multi)arc in $\mathrm{PHG}(R_R^t)$, where R is a chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$, and let x be a point with $\mathfrak{K}(x) = a$. Then*

$$k \le a + m_{n-a}(R_R^{t-1}).$$

Proof. Fix a hyperplane H with $x \not\frown H$. Define a projection φ from x onto H by

$$\varphi : \begin{cases} \mathcal{P} \setminus \{x\} & \rightarrow & H, \\ y & \rightarrow & \cup_{L: L \in \mathcal{L}, x \in L} L \cap H. \end{cases}$$

If $y \in \mathcal{P}$ is not a neighbour to x then its image is a point; if $y \frown x$ then the image of y is a neighbour class of points in $H \cong \mathrm{PHG}(R_R^{t-1})$. The image of an s-dimensional Hjelmslev subspace through x contains an $(s-1)$-dimensional subspace in H. Conversely, every $(s-1)$-dimensional subspace in H is contained in the projection of some s-dimensional subspaces of $\mathrm{PHG}(R_R^t)$ through x.

Define a new arc $\mathfrak{K}^{\varphi} : H \rightarrow \mathbb{Q}$ via

$$\mathfrak{K}^{\varphi}(z) := \sum_{y: \varphi(y)=z, y \not\frown x} \mathfrak{K}(y) + \frac{1}{q} \sum_{y: \varphi(y)=z, y \frown x} \mathfrak{K}(y).$$

Let H' be a hyperplane in $\mathrm{PHG}(R_R^t)$ containing x, and set $F' = \varphi(H')$. Then

$$\mathfrak{K}^{\varphi}(F') = \mathfrak{K}(H') - \mathfrak{K}(x).$$

Hence $\mathfrak{K}^{\varphi}$ is a $(k-a, n-a)$-arc with rational multiplicities of the points.

Now define

$$\varphi_0 : \begin{cases} \mathcal{P} \setminus \{x\} & \rightarrow & H, \\ y & \rightarrow & L \cap H, \end{cases}$$

where L is some arbitrarily chosen line in $\mathrm{PHG}(R_R^t)$ that contains x and y. Set

$$\mathfrak{K}^{\varphi_0}(z) := \sum_{y: \varphi_0(y)=z} \mathfrak{K}(y).$$

It is now easily verified that $\mathfrak{K}^{\varphi_0}(F') \leq \mathfrak{K}^{\varphi}(F')$. The arc $\mathfrak{K}^{\varphi_0}$ is integer-valued $(k-a, n-a)$-arc in H hence

$$n - a \leq m_{n-a}(R_R^{t-1}),$$

which had to be proved. $\square$

Corollary 2. *Let $\mathfrak{K}$ be a projective (k,n)-arc in $\mathrm{PHG}(R_R^4)$ where R is a chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$. Then*

$$k \leq 1 + m_{n-1}(R_R^3).$$

3 Arcs with multiple points in $\mathrm{PHG}(R_R^3)$

Extensive tables for the optimal sizes of projective arcs in Hjelmslev planes over the small chain rings are given in [1, 3, 5]. However, the arcs needed in Corollary 2 are not projective. In this section, we collect results on multiarcs of maximal size in projective Hjelmslev geometries of dimension 3. Let us note that the general bound from [5] applies also for arcs with multiple points. Since this bound is our main tool, we state it explicitly.

Theorem 3. *Let $\mathfrak{K}$ be a (k,n)-arc in $\mathrm{PHG}(R_R^3)$ where $|R| = q^2$, $R/N \cong \mathbb{F}_q$. Suppose there exist a point x with $\mathfrak{K}(x) = a$ and a neighbour class of points $[x]$ with $\mathfrak{K}([x]) = b$. Then*

$$k \leq (n-a)q^2 + (n-b)q + b.$$

For some special values of n, the exact value of $m_n(R_R^3)$ is easily found.

Theorem 4. *Let R be a chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$. Then*

(a) $m_{sq(q+1)}(R_R^3) = sq^2(q^2+q+1)$;

(b) $m_{sq(q+1)+1}(R_R^3) = sq^2(q^2+q+1)+1$.

Proof. Since part (a) follows easily from (b), we provide a proof for part (b) only.

Arcs with parameters $(sq^2(q^2+q+1)+1, sq(q+1)+1)$ are easily obtained as the sum of s copies of the complete plane plus an arbitrary point. Now we are going to prove that we cannot have an arc with $n = sq(q+1)+1$ and a larger size. Let $\mathfrak{K}$ be an (k,n)-with $n = sq(q+1)+1$.

Assume $[x]$ is a neighbour class of points with $\mathfrak{K}([x]) = sq^2 + \alpha$, $\alpha > 0$. There must be such a class; otherwise $k \leq sq^2(q^2+q+1)$ and we are done.

Now in each parallel class of $[x]$ (which has the structure of $\mathrm{AG}(2,q)$) we have a line segment of multiplicity at least $sq + \lceil \alpha/q \rceil$. For each line L having the direction of this line segment, we have

$$\mathfrak{K}(L \setminus [x]) \leq sq(q+1) - sq - \lceil \alpha/q \rceil = sq^2 + 1 - \lceil \alpha/q \rceil.$$

This implies

$$\begin{aligned} k &\leq sq^2 + \alpha + q(q+1)sq^2 + 1 - \lceil \alpha/q \rceil \\ &= sq^2(q^2+q+1) + q^2(1 - \lceil \alpha/q \rceil) + q(1 - \lceil \alpha/q \rceil) + \alpha. \end{aligned}$$

If $\alpha > q$, $q^2(1-\lceil \alpha/q \rceil) + q(1-\lceil \alpha/q \rceil) + \alpha < 0$. So let $1 \leq \alpha \leq q$. Clearly, there exists a point in $[x]$ of multiplicity $s+\beta$, $\beta \geq 1$, and at least one of the line segments through this point has multiplicity more than $sq+2$. Otherwise, we would have

$$\mathfrak{K}([x]) \leq s+\beta+(q+1)(sq+1-s-\beta) = sq^2+(q+1)(1-\beta) \leq sq^2,$$

a contradiction. Now

$$\begin{aligned} k &\leq sq^2+\alpha+q^2(sq^2+1-\lceil\alpha/q\rceil)+q(sq^2-\lceil\alpha/q\rceil \\ &= sq^2(q^2+q+1)+q^2(1-\lceil\alpha/q\rceil)-q\lceil\alpha/q\rceil)+\alpha. \end{aligned}$$

We have that $q^2(1-\lceil\alpha/q\rceil)-q\lceil\alpha/q\rceil)+\alpha \leq 1$. with equality for $\alpha=1$. This proves the theorem. □

The next theorem settles the problem of finding the maximal multiarcs for the rings with four elements.

Theorem 5. *Let R be a chain ring with $|R|=4$, $R/\operatorname{Rad} R \cong \mathbb{F}_2$. Then*

(i) $m_{6t}(R_R^3) = 28t$,

(ii) $m_{6t+1}(R_R^3) = 28t+1$,

(iii) $m_{6t+2}(R_R^3) = \begin{cases} 28t+7 & \text{if } R=\mathbb{Z}_4; \\ 28t+6 & \text{if } R=\mathbb{F}_2[X]/(X^2) \end{cases}$,

(iv) $m_{6t+3}(R_R^3) = 28t+10$,

(v) $m_{6t+4}(R_R^3) = 28t+16$,

(vi) $m_{6t+5}(R_R^3) = 28t+22$,

where $t=0,1,2,\ldots$.

Proof. Clearly, (i) and (ii) are settled by Theorem 4.

(iii) Let $n=6t+2$. Arcs of cardinality $28t+7$ (resp. $28t+6$) are obtained as a sum of T copies of the complete plane and a $(7,2)$-arc (resp. $(6,2)$-arc). Now assume there exists a (k,n)-arc with $k=28t+8$. Then there exists a class of points $[x]$ with $\mathfrak{K}([x])4t+2$ and a point in it with $\mathfrak{K}(x)\geq t+1$. By Theorem 3 we get

$$\begin{aligned} k &\leq 4\cdot(6t+2-a)+2\cdot(6t+2-b)+b \\ &= 36t+12-4a-b \\ &\leq 28t+6, \end{aligned}$$

which is a contradiction.

Now assume that $\mathfrak{K}$ is a (k,n)-arc with $k = 28t + 7$. By the above argument, all classes of points have multiplicity $4t + 1$ and every point has multiplicity at most $t + 1$. Moreover, the four points in each neighbour class of points must have multiplicities $t + 1, t, t, t$ since otherwise we get a contradiction by a counting argument. For instance, if the multiplicities of the points in a neighbour class are $t+1, t+1, t+1, t-2$ then there is a line segment of multiplicity $2t + 2$ in each direction and

$$k \leq 4t + 1 + 6 \cdot (6t + 2 - 2t - 2) = 28t + 1.$$

The other possibility of points of multiplicities $t + 1, t + 1, t, t - 1$ is ruled out by the same argument.

Now we have seven points of multiplicity $t + 1$ and obviously no three of them are collinear. Hence they form a $(7, 2)$-arc which is known to exist when $R = \mathbb{Z}_4$ and not to exist in case of $R = \mathbb{F}_2[X]/(X^2)$.

(iv) Obviously, we can construct $(28t + 10, 6t + 3)$-arcs as a sum of t copies of the plane and a $(10, 3)$-arc. Assume there is a (k,n)-arc $\mathfrak{K}$ with $k = 28t+11$ and $n = 6t+3$. Then there exists a class $[x]$ with $\mathfrak{K}([x]) \geq 4t+2$ and a point in it, x say, with $\mathfrak{K}(x) \geq 4t + 1$. By Theorem 3, $k \leq 28t + 12$. If $k = 28t + 11$ or $28t + 12$ then $\mathfrak{K}([x]) \leq 4t + 3$ for every class $[x]$ and $\mathfrak{K}(y) \leq t + 1$ for every point y in such a class.

Classes of multiplicity $4t+3$ are easily ruled out since they must consist of three $(t+1)$-points and one t-point, and must have segments of multiplicity $2t+2$ in every direction. By a similar argument, a class of multiplicity $4t+2$ consists of two $(t+1)$-points and two t-points. Now the $(t+1)$-points form a $(\kappa, 3)$-arc with $\kappa = 11$ or 12, which is impossible.

The proofs of (v) and (vi) use similar arguments. □

The next theorem gives better upper bounds for large values of n.

Theorem 6. *Let $\mathfrak{K}$ be a (k,n)-arc in $\mathrm{PHG}(R_R^4)$, where R is a chain ring with $|R| = q^2$, $R/\operatorname{Rad} R \cong \mathbb{F}_q$. If there exists a neighbour class of lines $[L]$ with $\mathfrak{K}([L]) \geq c$ then*

$$k \leq q(q+1)(n - \lceil \frac{c}{q} \rceil) + c.$$

Remark 3.1. There exists a spread of $q^2 + 1$ lines of the factor geometry $\mathrm{PG}(3, q)$. Hence we have the estimate $c \geq \lceil k/(q^2 + 1) \rceil$. Now applying Theorem 6, one gets that for every (k,n)-arc in $\mathrm{PHG}(R_R^4)$ the following inequality holds:

$$k - \lceil \frac{k}{q^2+1} \rceil + q(q+1) \lceil \frac{1}{q} \lceil \frac{k}{q^2+1} \rceil \rceil \leq nq(q+1).$$

4 Arcs in three-dimensional Hjelmslev spaces

In this section, we present a table with exact values and bounds on $m_n(R^4_R)$, $n \leq 28$, for the two chain rings with four elements. Let us note that for $n = 3, 4, 5$ the exact values are computed in [4]. For $n \geq 20$, the constructions are obtained by deleting disjoint lines from the geometry $\mathrm{PHG}(R^3_R)$. Note that there exists a line spread of $\mathrm{PHG}(R^4_R)$. Hence we can select any number (up to $q(q^2+1)$) of disjoint lines.

In some cases we can get larger arcs. Let us note that a subgeometry isomorphic to $\mathrm{PG}(3, q)$ blocks each plane at least $q + 1$ times. Such a subgeometry exists if the underlying chain ring R contains a subring isomorphic to the factor field. Moreover, there exists a partition of $\mathrm{PHG}(R^4_R)$ in three dimensional projective spaces of order q. Hence in the geometry over $R = \mathbb{F}_q[X; \sigma]/(X^2)$, we have blocking sets with parameters $(15, 3)$, $(30, 6)$ and $(45, 9)$. These blocking sets produce arcs with parameters $(105, 25)$, $(90, 22)$ and $(75, 19)$.

The arcs with $6 \leq n \leq 15$ are produced as a sum of suitable plane arcs.

Table 1. Values of $m_n(R^4_R)$ for Hjelmslev planes of order $q^2 = 4$

n/R	$\mathbb{Z}_4$	$\mathbb{F}_2[X]/(X^2)$	n/R	$\mathbb{Z}_4$	$\mathbb{F}_2[X]/(X^2)$
3	8	6	16	64 – 67	64 – 67
4	10	11	17	65 – 70	65 – 70
5	16	16	18	66 – 76	66 – 76
6	18 – 23	18 – 23	19	67 – 80	75 – 80
7	21 – 29	19 – 29	20	72 – 83	76 – 83
8	22 – 30	22 – 30	21	78 – 90	78 – 90
9	24 – 36	24 – 35	22	84 – 91	90 – 91
10	26 – 39	26 – 39	23	90 – 98	90 – 98
11	30 – 45	30 – 45	24	96 – 104	96 – 104
12	36 – 51	36 – 51	25	102 – 105	105
13	40 – 57	40 – 57	26	108 – 110	108 – 110
14	44 – 58	44 – 58	27	114	114
15	50 – 61	50 – 61	28	120	120

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