

On weight distributions of perfect structures

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Abstract. We study the weight distribution of a perfect coloring (equitable partition) of a graph with respect to a completely regular code (in particular, with respect to a vertex if the graph is distance-regular). We show how to compute this distribution by the knowledge of the percentage of the colors over the code. For some partial cases of completely regular codes we derive explicit formulas of weight distributions. Since any (other) completely regular code itself generates a perfect coloring, this gives universal formulas for calculating the weight distribution of any completely regular code from its parameters.

1 Perfect colorings and completely regular codes

Let $G = (V = \{0, \dots, N-1\}, E)$ be a graph; let f be a function (“coloring”) on V that possesses exactly k different values e_0, \dots, e_{k-1} (“colors”). The function f is called a *perfect coloring* with parameter matrix $S = (S_{ij})_{i,j=1}^k$, or *S -perfect coloring*, iff for any i, j from 0 to $k-1$ any vertex of color e_i has exactly S_{ij} neighbors of color e_j . (The corresponding partition of V into k parts is known as an *equitable partition*. In another terminology, see e.g. [3], the coloring f is called *S -feasible* and S is called a *front divisor* of G .)

In what follows we assume that e_i is the tuple with 1 in the i th position and 0s in the others (the length of the tuple may vary depending on the context; in the considered case it is k). Denote by A the adjacency matrix of G . Then it is easy to see [7, Lemma 9.3.1] that f is an S -perfect coloring if and only if

$$Af = fS \tag{1}$$

where the function f is represented by its value array; i.e., the i th row of the $|V| \times k$ matrix f is $f(i)$. If the equation (1) holds for some matrices A , S , and f (of size $N \times N$, $k \times k$, and $N \times k$ respectively) over R , then we will say that f is an *S -perfect structure* (or a perfect structure with parameters S) over A [2].

So, in this context the concept of perfect structure is a continuous generalization of the concept of perfect coloring. Conversely, a perfect coloring (equitable partition) is equivalent to a perfect structure over the graph (i.e., over its adjacency matrix) with the rows from e_0, \dots, e_{k-1} .

Suppose that f satisfying (1) is three-diagonal. In this case, the corresponding perfect coloring (if any) has the following property: the colors e_i, e_j of any two neighbor vertices satisfy $|i - j| \leq 1$. The support of the e_0 of such coloring is known as a *completely regular code* with covering radius $k - 1$. In other words, a set C of vertices of a graph $G = (V, E)$ is a *completely regular code* if and only if its *distance coloring* (i.e., the function $f(x) = e_{d_G(x, C)}$ where $d_G(\cdot, \cdot)$ is the natural distance in the graph) is perfect.

Example 1 Very partial, but also very important, case of perfect structures is the case of $k = 1$; then f is just an eigenvector of A (in the graph case, an eigenfunction of the graph) with the eigenvalue equal to the only element of S .

Example 2 A subset P of the vertex set V of a regular graph $G = (V, E)$ is known to be a *1-perfect code* if its distance coloring is perfect with the parameter matrix $((0, d), (1, d - 1))$, where d is the degree of the graph. 1-Perfect codes in n -cubes (see Example 3 below) are actively studied; the best-investigated case is binary, see e.g. [8, 12], but even in that case the problem of full characterization of such codes is far from being solved.

2 Distributions

Assume that we have two perfect structures f and g over A with parameters S and R respectively. Then we say that $g^T f$ is the *distribution of f with respect to g* . For perfect colorings, this has the following sense: the element in the i th column and j th row of $g^T f$ equals the number of the vertices v such that $g(v) = e_i$ and $f(v) = e_j$ (to avoid misunderstanding, we note that the number of elements in e_i in the first equation is the number of colors of the coloring g , while the number of elements in e_j in the last equation is the number of colors of f ; in general, these numbers can be different, even if $i = j$). In the case when g is the distance coloring of some set (completely regular code) C , we will also say that $g^T f$ is the *weight distribution of f with respect to C* (if $C = \{c\}$, with respect to c). In other words the weight distribution of f with respect to C is the tuple $(f_0, f_1, f_2, \dots)^T$ where f_w is the sum of f over the all vertices at the distance w from C .

The two following statements are elementary from the algebraic point of view; nevertheless, they are very significant for the perfect structures.

Theorem 1 *Let that f and g be S - and R - perfect structures over A and A^T respectively (f, g, A, S , and R are $N \times k$, $N \times m$, $N \times N$, $k \times k$, and $m \times m$ matrices). Then $g^T f$ is a perfect structure over R^T with parameters S .*

Note that if A is the adjacency matrix of some graph, then $A = A^T$.

Theorem 2 *If the matrix $B = \{B_{i,j}\}_{i,j=0}^{n-1}$ satisfies $B_{i,i+1} \neq 0$, $B_{i,j} = 0$ for any $i = 0, \dots, n-2$, $j > i+1$, then any perfect S -structure f over B is uniquely defined by its first row f_0 . Moreover, the rows f_i satisfy the recursive relations*

$$f_i = (f_{i-1}S - \sum_{j=0}^{i-1} B_{i,j}f_j)/B_{i-1,i}, \quad i = 1, \dots, n-1, \quad (2)$$

and, as follows, $f_i = f_0\Pi_i^{(B)}(S)$ where $\Pi_i^{(B)}(x)$ is a degree- i polynomial in x .

So, given a completely regular code C , we also have a way to reconstruct the weight distribution with respect to C of any other perfect structure (perfect coloring) f over the same graph by knowledge of only the first component the distribution (the sum of the function f over the set C). To do this, we should apply Theorem 2 with $B = R^T$, where the three-diagonal matrix R is the parameters of the distance coloring of C . The uniqueness of such the reconstruction was known before [1], but known formulas cover only partial cases of f (e.g., the weight distribution of 1-perfect binary codes [9, 11]).

3 Weight distributions in distance-regular graph

Let $G = (V, E)$ be a graph and let for every w from 0 to $\text{diameter}(G)$ the matrix $A_w^{(G)} = A_w = (a_{ij}^w)_{i,j \in V}$ be the distance- w matrix of G (i.e., $a_{ij}^w = 1$ if the graph distance between i and j is w , and $= 0$ otherwise); put $A := A_1$. The graph G is called *distance regular* iff for every w the matrix A_w equals $\Pi_w(A)$ for some polynomial Π_w of degree w . The polynomials $\Pi_0, \Pi_1, \dots, \Pi_{\text{diameter}(G)}$ are called *P -polynomials* of G .

Now suppose that f is a perfect structure over G (i.e., over A) with some parameters S . By the definition, we have $Af = fS$. From this we easily derive $A^w f = A^{w-1}fS = \dots = fS^w$ for any degree w and, consequently, $P(A)f = fP(S)$ for any polynomial P . In particular, we get

$$A_w f = f\Pi_w(S). \quad (3)$$

We now observe that the i th row of $A_w f$ is the sum of the vector-function f over the all vertices at the distance w from the i th vertex. So, for perfect colorings (3) means the following:

Theorem 3 *Assume we have an S -perfect coloring of a distance-regular graph with P -polynomials Π_w . If the color of the vertex x is e_j , then the percentage of the colors at the distance w from x is calculated as $e_j\Pi_w(S)$.*

In other words, the weight distribution of the perfect coloring with respect to any color- e_j vertex is $(e_j = e_j\Pi_0(S), e_j\Pi_1(S), \dots, e_j\Pi_{\text{diameter}(G)}(S))^T$.

Example 3 Let $G = H_q^n$ be the q -ary n -cube, whose vertex set is the set of all n -words over the alphabet $\{0, \dots, q-1\}$, two vertices being adjacent if and only if they differ in exactly two positions. Then

$$\Pi_w(\cdot) = P_w(P_1^{-1}(\cdot)) \quad (4)$$

where

$$P_w(x) = P_w(x; n, q) = \sum_{j=0}^w (-1)^j (q-1)^{w-j} \binom{x}{j} \binom{n-x}{w-j} \quad (5)$$

is the *Krawtchouk polynomial*; $P_1(x) = (q-1)n - qx$.

Example 4 Any of the two connected components H_+^n, H_-^n of the distance-2 graph of H_2^n (two different vertices are adjacent in the distance-2 graph if and only if they have a common neighbor in H_2^n) is also a distance-regular graph, which is known as a *halved n -cube*. Since $A_w^{(H_+^n \cup H_-^n)} = A_{2w}^{(H_2^n)}$, for H_+^n and H_-^n we have $\Pi_w(\cdot) = P_{2w}(P_2^{-1}(\cdot))$ where $P_w(x) = P_w(x; n, 2)$ is the Krawtchouk polynomial (5); $P_2(x) = \binom{n}{2} - 2nx + 2x^2$.

Example 5 Let $G = J(n, k)$ be the *Johnson graph*, whose vertex set is the set of all binary n -tuples with exactly k ones, two vertices being adjacent if and only if they differ in exactly two positions. Then $\Pi_w(\cdot) = E_w(E_1^{-1}(\cdot))$ where

$$E_w(x) = E_w(x; n, k) = \sum_{j=0}^w (-1)^j \binom{x}{j} \binom{k-x}{w-j} \binom{n-k-x}{w-j}$$

is the *Eberlein polynomial* [4].

4 Distributions with respect to some sets

In this section we will derive formulas for the weight distributions of perfect structures with respect to some special completely regular codes, which have large covering radius and small (1 or 2) code distance. As we will see, for the considered cases the situation is reduced to calculating the weight distributions with respect to a vertex in some smaller distance-regular graph.

4.1 A lattice

The set R discussed in this subsection plays some role in the theory of perfect colorings of the n -cube. It occurs in constructions of perfect colorings [5, 6]; it necessarily occurs in any linear distance-2 completely regular binary code [13]; in particular, in shortened 1-perfect binary code, and a (non-shortened)

variation of this set (case $m = 2$), known as a linear i -component, is widely used for the construction of 1-perfect binary codes, see e.g. [12]. We will derive a rather simple formula for the weight distribution of a perfect coloring with respect to R .

Let us consider the q -ary mk -cube H_q^{mk} and the function $\tilde{g} : V(H_q^{mk}) \rightarrow V(H_q^k)$ defined as

$$\tilde{g}(x_1, \dots, x_m) = x_1 + \dots + x_m \bmod q, \quad x_i \in H_q^k. \quad (6)$$

The set R is defined as the set of zeroes of \tilde{g} .

Lemma 1 *\tilde{g} is a perfect coloring of H_q^{mk} with the matrix $mA^{(H_q^k)}$ where $A^{(H_q^k)}$ is the adjacency matrix of H_q^k .*

So, after representing the values of \tilde{g} by the corresponding tuples $e_i \in \{0, 1\}^{V(H_q^k)}$

$$g(x) = e_{\tilde{g}(x)},$$

we have the equation

$$A^{(H_q^{mk})}g = gmA^{(H_q^k)}.$$

By Theorem 1, for any other perfect structure f over H_q^{mk} with parameters S we have $(mA^{(H_q^k)})(g^T f) = (g^T f)S$ or, equivalently,

$$A^{(H_q^k)}(g^T f) = (g^T f)\left(\frac{1}{m}S\right).$$

I.e., $(g^T f)$ is a perfect structure over H_q^k with parameters $\frac{1}{m}S$. Taking into account the following simple fact, we see that our problem is reduced to the calculation of the weight distribution of this new perfect structure with respect to the zero vertex.

Lemma 2 *The distance from a vertex x to R coincides with the distance from $\tilde{g}(x)$ to the zero.*

So, we can use the results of the previous section to calculate the weight distribution of f with respect to R .

Theorem 4 *Let f be an S -perfect structure over the q -ary mk -cube H_q^{mk} . Let f_0 be the sum of f over the set R of zeroes of \tilde{g} (6). Then the weight distribution of f with respect to R is*

$$(f_0 = f_0\Pi_0(\frac{1}{m}S), f_0\Pi_1(\frac{1}{m}S), \dots, f_0\Pi_k(\frac{1}{m}S))^T,$$

where the polynomials Π_i are defined by (4) and (5).

4.2 The direct product

Here we consider distributions with respect to an instance of one of the multipliers in the direct product of two graphs. Given two graphs $G' = (V', E')$, $G'' = (V'', E'')$, their direct product $G' \times G'' = (V, G)$ is defined as follows: the vertex set is the set $V' \times V'' = \{(v', v'') : v' \in V', v'' \in V''\}$; two vertices $u = (u', u'')$ and $v = (v', v'')$ are adjacent (i.e. $\{u, v\} \in E$) if and only if either $u' = v'$ and $\{u'', v''\} \in E''$ or $\{u', v'\} \in E'$ and $u'' = v''$.

Theorem 5 *Let f be an S -perfect structure over the direct product $G' \times G''$ of a regular graph $G' = (V', E')$ and a distance-regular graph $G'' = (V'', E'')$. Let F be some instance of G' in $G' \times G''$, i.e., the subgraph of $G' \times G''$ generated by the vertex subset $V' \times \{o\}$ for some $o \in V''$. Let f_0 be the sum of f over F . Then the weight distribution of f with respect to F is*

$$(f_0 = f_0 \Pi_0(S - dI), f_0 \Pi_1(S - dI), \dots, f_0 \Pi_{\text{diameter}(G'')}(S - dI))^T,$$

where d is the degree of G' and Π_0, Π_1, \dots are the P -polynomials of G'' .

A known example of the graph direct product is the q -ary n -cube H_q^n , which is the direct product of n copies of the full graph K_q . For any integer m from 0 to n we have $H_q^n = H_q^m \times H_q^{n-m}$, and Theorem 5 means the following.

Corollary 1 *Let f be an S -perfect structure over the q -ary $(m+k)$ -cube H_q^{m+k} . Let F be a subcube of H_q^{m+k} of dimension m (isomorphic to H_q^m), and let f_0 be the sum of f over F . Then the weight distribution of f with respect to F is*

$$(f_0 = f_0 \Pi_0(S'), f_0 \Pi_1(S'), \dots, f_0 \Pi_m(S'))^T,$$

where $S' = S - (q-1)mI$, $\Pi_w(\cdot) = P_w(P_1^{-1}(\cdot))$, and $P_w(x) = P_w(x; k, q)$ are the Krawtchouk polynomial (5).

4.3 A subcube of smaller size

Here we consider distributions with respect to another completely regular code in H_q^n with large covering radius, the subcube H_p^n of the same dimension and a smaller order $p < q$. Note that if p divides q , then the theorem can be proved using the approach of the previous two subsections; in the general case we can calculate the parameters of the distance coloring and use the known recursive formulas for the Krawtchouk polynomials (see, e.g., [10, § 5.7]).

Theorem 6 *Let f be an S -perfect structure over the q -ary n -cube H_q^n . Let $p < q$, and let f_0 be the sum of f over an p -ary subcube $H_p^n \subset H_q^n$. Then the weight distribution of f with respect to H_p^n is*

$$(f_0 = f_0 \Pi_0(S'), f_0 \Pi_1(S'), \dots, f_0 \Pi_k(S'))^T,$$

where $S' = (S - (p-1)nI)/p$, $\Pi_i(\cdot) = P_i(P_1^{-1}(\cdot))$, $P_i(x) = P_i(x; n, \frac{q}{p})$ (5).

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