Sixth International Workshop on Optimal Codes and Related Topics
June 16-22, 2009, Varna, Bulgaria *pp. 114-116*

Nonadaptive search for two elements with sets of equal sum

Emil Kolev emil@math.bas.bg
Institute of Mathematica and Informatics
Bulgarian Academy of Sciences

Abstract. In this note we find all "good" values S for which two unknown elements from the set $\{1, 2, 3, \ldots, 2^n\}$, $n \geq 4$, can be found by questions $B \subset A$ such that the sum of the elements of B equals S.

1 Introduction

Consider the set $A = \{1, 2, 3, \ldots, 2^n\}$ and assume two of its elements, say x and y, are being marked. For a given positive integer S a set $B \subset A$ is permissible if the sum of its elements equals S. We are allowed to choose a permissible set B and ask whether B contains a marked element or not. The answer is 0 if non of x and y lies in B, 1 if only one of them is in B and 2 if both of them are in B. We find all values of S for which it is possible to find x and y using nonadaptive search. The same search problem for one unknown element has been discussed in [1, 2, 3]. For other search problems see [4, 5, 6].

As usual each subset B of A is represented by the *characteristic* vector $V = (v_1, v_2, \ldots, v_{2^n})$ where $v_i = 1$ if $i \in B$ and $v_i = 0$ otherwise. It is clear that the scalar product of a characteristic vector for permissible set with $(1, 2, \ldots, 2^n)$ equals S.

Further, consider a collection of permissible sets $B_1, B_2, \ldots, B_k$. Define $k \times 2^n$ *characteristic* matrix G which rows are all characteristic vectors of $B_1, B_2, \ldots, B_k$. Let $a = (a_1, a_2, \ldots, a_k)^t$ be the vector of the corresponding answers. If x and y are uniquely determined by these answers then the sum of x-th and y-th columns of G equals a^t and there are is no other pair of columns with the same property.

Therefore in order to find x and y we need to find a binary matrix G such that the scalar product of any row of G with $(1, 2, \ldots, 2^n)$ equals S and no two pairs of columns have one and the same sum. We find all integers S for which such matrix exists. It turns out the result is the same as for the case of one unknown element (see [1]).

2 Main result

In this section we find all good values of S.

Theorem 1. An integer S is "good" if and only if

$$S \in \left[2^n - 1, 2^{2n-1} - 2^{n-1} + 1\right].$$

Proof. Consider all permissible sets and find the corresponding characteristic matrix.

First, we show that this matrix contains no equal columns. It suffices to show that for any two integers a and b, $a < b$ from A there exists permissible set B for which $a \notin B$ and $b \in B$ or $a \in B$ and $b \notin B$. Apply greedy algorithm to express $S - b$ as a sum of elements from $A \backslash \{a, b\}$. If it succeeds then the desired set exists. When $a \geq 4$ we have that $a = 1 + (a - 1)$ and $b = 1 + (b - 1) = 2 + (b - 2)$ and therefore the greedy always succeeds. When $a = 3$ and $b \geq 5$ we have $a = 1 + 2$ and $b = 1 + (b - 1)$ and the greedy can be completed. If $a = 3$ and $b = 4$ then the algorithm fails only if at some point we need 4 to complete the algorithm. Note that since $S \leq 2^{2n-1} - 2^{n-1} + 1$ we have that 5 is not a term in the sum so far. Now replace $4 + 4$ by $3 + 5$ and we obtain permissible set for which $a \in B$ and $b \notin B$. The cases $a = 2$ and $a = 1$ are treated in similar manner.

Second, we prove that there are no two pairs of columns of the characteristic matrix having one and the same sum. It suffices to show that for any $a, b, c, d \in A$, $a < b < c < d$ there exists permissible set B for which $|B \cap \{a, b, c, d\}|$ is odd. Apply greedy algorithm to express $S - b - c - d$ as a sum of elements from the set $A \backslash \{a, b, c, d\}$. If it succeeds we have $|B \cap \{a, b, c, d\}| = 3$ and we are done. Suppose $a \geq 3$, $b - a \geq 2$, $c - b \geq 2$ and $d - c \geq 2$. For $x \in \{a, b, c, d\}$ we have $x = 1 + (x - 1)$ and $1, x - 1 \notin \{a, b, c, d\}$, so the greedy algorithm works.

All the remaining cases are considered separately. The observations are long and somehow tedious. For example, let $a = 2$, $b - a \geq 2$, $c - b = 1$ and $d - c \geq 2$. Since $b = 1 + (b - 1)$, $d = 1 + (d - 1)$ the greedy algorithm could fail only when at some point we are left with a or c to complete the sum. Suppose first this is c. Since for $c \geq 7$ we have $c = 3 + (c - 3)$ where $\{a, b\} \cap \{3, c - 3\} = \emptyset$ it follows that greedy always works when $c \geq 7$. It remains to consider the cases $c = 6$ or $c = 5$. If $d - c \geq 3$ then we replace $b + 2c = 5 + 2.6 = 17$ by $7 + 8 + 1 + 4$ (when $c = 6$) and $b + 2c = 4 + 2.5 = 14$ by $6 + 7 + 1$ (when $c = 6$). In both cases we are done. When $d - c = 1$ we have $\{a, b, c, d\} = \{2, 5, 6, 8\}$ or $\{a, b, c, d\} = \{2, 4, 5, 7\}$. In both cases it follows from $S \leq 2^{2n-1} - 2^{n-1} + 1$ that $d + 1$ is not a term in the sum obtained by the greedy algorithm. Since $b + 2c = 17 = 1 + 3 + 4 + 9$ for the first case and $b + 2c = 4 + 2.5 = 6 + 8$ for the second one, the greedy works.

Now, assume the algorithm stops at $a = 2$. Observe that $c - 2 \neq 2$ and $c - 2 \neq b$. If we replace c by 2 and $c - 2$ we again find permissible set B such that $|B \cap \{a, b, c, d\}| = 3$. $\diamond$

Let S be an integer from the interval from Theorem 1. To determine the minimum number of questions needed to find the two unknown elements seems difficult task. An easy lower bound is given by

Proposition 1. If k is the minimum number of questions needed to find the two unknown elements then

$$3^k - 2^k \geq 2^{2n-1} - 2^{n-1}.$$

Acknowledgments. This research was partially supported by the Bulgarian NSF under Contract MM-1405/04.

References

[1] E. Kolev, Nonadaptive search with sets of given sum, *Proc. ACCT8*, Tsarskoe selo, 2002, 159-162.

[2] E. Kolev, Nonadaptive search problem with sets of equal sum, *CEJM* 1, 2003, 272-283.

[3] E. Kolev, Nonadaptive search problem in weighted set, *Proc. ACCT10*, Zvenigorod, Russia, 2006.

[4] R. Hill, J. P.Karim, Searching with lies: the Ulam problem, *Discr. Math.* 106-107, 1992, 273-283.

[5] E. Kolev, I. Landgev, On a two-dimensional search problem, *Serdica Math. J.* 21, 1995, 219-230.

[6] M. Ruszinko, On a 2- and 3-dimensional search problem, *Proc. Sixth Swedish-Russian Workshop Inform. Theory*, Aug. 21-27, 1993. Mölle, 437-440.