Sixth International Workshop on Optimal Codes and Related Topics
June 16-22, 2009, Varna, Bulgaria *pp. 102-105*

New bounds for the length of the optimal (N,11,2,3) superimposed codes[1]

Stoyan Kapralov s.kapralov@gmail.com
Department of Mathematics, Technical University of Gabrovo
Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
BULGARIA

Mladen Manev ml.manev@gmail.com
Department of Mathematics, Technical University of Gabrovo, BULGARIA

Abstract. We prove that $36 \le N(11, 2, 3) \le 40$.

1 Introduction and Preliminaries

Definition 1 *A binary $N \times T$ matrix $C = (c_{ij})$ is called an (N, T, w, r) superimposed code (SIC) if for any pair of subsets $W, R \subset \{1, 2, \dots, T\}$ such that $|W| = w$, $|R| = r$ and $W \bigcap R = \emptyset$, there exists a coordinate $i \in \{1, 2, \dots, N\}$ such that $c_{ij} = 1$ for all $j \in W$ and $c_{ij} = 0$ for all $j \in R$.*

Let $N(T, w, r)$ be the minimum length N for which an (N, T, w, r) SIC exists for fixed values of T, w and r. The exact values of $N(T, 2, 3)$ are known only for $T \le 10$. Recently Manev proved that $N(8, 2, 3) = 8$, and $N(9, 2, 3) = 30$ [3]. It is known that $N(10, 2, 3) = 30$ and $33 \le N(11, 2, 3) \le 45$ [2].

In this article we study the value of $N(11, 2, 3)$ and we prove that $36 \le N(11, 2, 3) \le 40$.

Definition 2 *Let x be a column of the superimposed code C. The residual code $Res(C, x = a)$ is the code obtained in the following way:*
1) take the i^{th} row $(i = 1, 2, ..., N)$ iff $c_{ix} = a$;
2) delete the column x in the selected rows.

In case C is an (N, T, w, r) SIC it is obvious that the residual code $Res(C, x = 0)$ is an $(N_0, T - 1, w, r - 1)$ SIC, while $Res(C, x = 1)$ is an $(N_1, T - 1, w - 1, r)$ SIC.

Definition 3 *Two (N, T, w, r) superimposed codes are equivalent if one of them can be obtained from the other by a permutation of the rows and a permutation of the columns.*

[1]Partially supported by the Technical University of Gabrovo under Grant C-901/2009.

2 Upper bound

Lemma 4 *If there exists an* $(N_0, T-1, 2, 3)$ *SIC, and there exists an* $(N_1, T-1, 1, 3)$ *SIC, then there exists an* $(N_0 + N_1, T, 2, 3)$ *SIC.*

Proof Let C_0 be a matrix of an $(N_0, T-1, 2, 3)$ SIC, and C_1 be a matrix of an $(N_1, T-1, 1, 3)$ SIC. Then the matrix

$$\left(\begin{array}{c|c} 0 & \\ \vdots & C_0 \\ 0 & \\ \hline 1 & \\ \vdots & C_1 \\ 1 & \end{array}\right)$$

is an $(N_0 + N_1, T, 2, 3)$ SIC. □

Corollary 5 $N(T, 2, 3) \leq N(T-1, 2, 3) + N(T-1, 1, 3)$

Corollary 6 $N(11, 2, 3) \leq 40$.

Proof Follows from the known results $N(10, 2, 3) = 30$ and $N(10, 1, 3) = 10$ [2]. □

3 Lower bound

Lemma 7 *If* C *is a* $(35, 11, 2, 3)$ *superimposed code then* $10 \leq wt(x) \leq 15$ *for any column* x *of* C*, where* $wt(x)$ *is the Hamming weight of* x.

Proof The residual code $Res(C, x = 0)$ is an $(N_0, 10, 2, 2)$ SIC. It is known that $N(10, 2, 2) = 20$, consequently $wt(x) \leq 15$. The residual code $Res(C, x = 1)$ is an $(N_1, 10, 1, 3)$ SIC. It is known that $N(10, 1, 3) = 10$, consequently $wt(x) \geq 10$. □

Lemma 8 *There is no* $(35, 11, 2, 3)$ *superimposed code with a column of weight* 10*,* 11 *or* 12*.*

Proof Suppose C is a $(35, 11, 2, 3)$ SIC and x is a column of weight v =10,11 or 12. Then the residual code $Res(C, x = 1)$ is a $(v, 10, 1, 3)$ SIC.

The identity matrix is the unique $(10, 10, 1, 3)$ SIC.

We classified (up to equivalence) all $(11, 10, 1, 3)$ and $(12, 10, 1, 3)$ SIC by a computer program. It turned out that each of the constructed codes contains as a submatrix the identity matrix of order 10.

If we delete the column x and all the rows with weight less or equal to 2 in the code C we obtain an $(N', 10, 2, 3)$ SIC with $N' \leq 25$. This is a contradiction because $N(10, 2, 3) = 30$. □

Lemma 9 *There is no* $(35, 11, 2, 3)$ *superimposed code with a column of weight* 15.

Proof By Lemma 8 the possible weights of a column in a $(35, 11, 2, 3)$ SIC are 13, 14 and 15. Suppose C is a $(35, 11, 2, 3)$ SIC and x is a column of weight 15. Then the residual code $Res(C, x = 0)$ is a $(20, 10, 2, 2)$ SIC. There is unique $(20, 10, 2, 2)$ SIC [1]. So the matrix of the code C is of the form:

$$\left(\begin{array}{c|c} 0 & \\ \vdots & C_0 \\ 0 & \\ \hline 1 & \\ \vdots & C_1 \\ 1 & \end{array}\right)$$

where C_0 is the unique $(20, 10, 2, 2)$ SIC or its complementary code.

Using an exhaustive computer search we tried to construct the matrix C_1. It turned out that the extension is impossible. □

Lemma 10 *There is no* $(35, 11, 2, 3)$ *superimposed code with a column of weight* 14.

Proof It follows from Lemmas 8 and 9 that the possible weights of a column in a $(35, 11, 2, 3)$ SIC are 13 and 14. Suppose C is a $(35, 11, 2, 3)$ SIC and x is a column of weight 14. Then the residual code $Res(C, x = 0)$ is a $(21, 10, 2, 2)$ SIC and the structure of the $(35, 11, 2, 3)$ code matrix is similar to that in Lemma 9. We classified up to equivalence all $(21, 10, 2, 2)$ SIC and obtained that there are 72 possibilities for the matrix C_0. For each of these cases we tried to construct C_1-part. It turned out, however, that there is no solution. □

Lemma 11 *There is no* $(35, 11, 2, 3)$ *superimposed code with a column of weight* 13.

Proof Suppose C is a $(35, 11, 2, 3)$ SIC. It follows from Lemmas 8, 9 and 10 that all columns of C are of weight 13.

Up to equivalence we may assume that the code C has the form similar to that of Lemma 9 where the matrix C_0 is a $(22, 10, 2, 2)$ SIC, and the matrix C_1 is a $(13, 10, 1, 3)$ SIC. Not every $(13, 10, 1, 3)$ SIC could be places at the C_1-part. The C_1 SIC must have at most 5 row of weight less than 2. Else we could take the matrix $\binom{C_0}{C_1}$ and delete the rows of weight less than 2. The remaining matrix would be an $(N', 10, 2, 3)$ SIC with $N' < 30$ – a contradiction. We obtained that there are exactly 7 inequivalent possibilities for the matrix C_1. The weight distributions of the columns are the following (B_i is the number of columns of weight i):

C_1	B_1	B_2	B_3	B_4
$1, 2, 3, 4$				10
5			1	9
6		1		9
7	1			9

Consequently there are exactly 4 possibilities for the weight distribution of the code C_0. Using an exhaustive computer search we found 6 inequivalent $(22, 10, 2, 2)$ codes with these weight distributions:

C_0	B_{12}	B_{11}	B_{10}	B_9
$1, 2, 3, 4$				10
5			1	9
6		1		9

Then we tried to assemble the two parts, $(22, 10, 2, 2)$ and $(13, 10, 1, 3)$ superimposed codes, in such a way that the whole matrix to be a $(35, 11, 2, 3)$ SIC. It turned out, however, that there is no solution.

$\square$

In this way we proved the main result of this section:

Theorem 12 *There is no* $(35, 11, 2, 3)$ *superimposed code.*

Corollary 13 $N(11, 2, 3) \geq 36$.

References

[1] S. Kapralov, M. Manev, The uniqueness of the optimal (20,10,2,2) superimposed code, *Proc. 35th Spring Conf. Union Bulg. Math.*, Borovets, Bulgaria, Apr. 5-8, 2006, 265-269.

[2] H. K. Kim, V. S. Lebedev, On optimal superimposed codes, *J. Combin. Designs*, 12, 2004, 79-91.

[3] M. Manev, The uniqueness of the optimal (28,8,2,3) and (30,10,2,3) superimposed codes, this volume, 151-156.