New bounds for the length of the optimal (N,11,2,3) superimposed codes¹

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Abstract. We prove that $36 \le N(11, 2, 3) \le 40$.

1 Introduction and Preliminaries

Definition 1 A binary $N \times T$ matrix $C = (c_{ij})$ is called an (N, T, w, r) superimposed code (SIC) if for any pair of subsets $W, R \subset \{1, 2, ..., T\}$ such that |W| = w, |R| = r and $W \cap R = \emptyset$, there exists a coordinate $i \in \{1, 2, ..., N\}$ such that $c_{ij} = 1$ for all $j \in W$ and $c_{ij} = 0$ for all $j \in R$.

Let N(T, w, r) be the minimum length N for which an (N, T, w, r) SIC exists for fixed values of T, w and r. The exact values of N(T, 2, 3) are known only for $T \leq 10$. Recently Manev proved that N(8, 2, 3) = 8, and N(9, 2, 3) = 30 [3]. It is known that N(10, 2, 3) = 30 and $33 \leq N(11, 2, 3) \leq 45$ [2].

In this article we study the value of N(11, 2, 3) and we prove that $36 \le N(11, 2, 3) \le 40$.

Definition 2 Let x be a column of the superimposed code C. The residual code Res(C, x = a) is the code obtained in the following way: 1) take the i^{th} row (i = 1, 2, ..., N) iff $c_{ix} = a$; 2) delete the column x in the selected rows.

In case C is an (N, T, w, r) SIC it is obvious that the residual code Res(C, x = 0) is an $(N_0, T - 1, w, r - 1)$ SIC, while Res(C, x = 1) is an $(N_1, T - 1, w - 1, r)$ SIC.

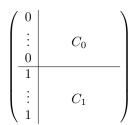
Definition 3 Two (N, T, w, r) superimposed codes are equivalent if one of them can be obtained from the other by a permutation of the rows and a permutation of the columns.

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2 Upper bound

Lemma 4 If there exists an $(N_0, T-1, 2, 3)$ SIC, and there exists an $(N_1, T-1, 1, 3)$ SIC, then there exists an $(N_0 + N_1, T, 2, 3)$ SIC.

Proof Let C_0 be a matrix of an $(N_0, T - 1, 2, 3)$ SIC, and C_1 be a matrix of an $(N_1, T - 1, 1, 3)$ SIC. Then the matrix



is an $(N_0 + N_1, T, 2, 3)$ SIC.

Corollary 5 $N(T,2,3) \le N(T-1,2,3) + N(T-1,1,3)$

Corollary 6 $N(11, 2, 3) \le 40.$

Proof Follows from the known results N(10, 2, 3) = 30 and N(10, 1, 3) = 10 [2].

3 Lower bound

Lemma 7 If C is a (35, 11, 2, 3) superimposed code then $10 \le wt(x) \le 15$ for any column x of C, where wt(x) is the Hamming weight of x.

Proof The residual code Res(C, x = 0) is an $(N_0, 10, 2, 2)$ SIC. It is known that N(10, 2, 2) = 20, consequently $wt(x) \le 15$. The residual code Res(C, x = 1) is an $(N_1, 10, 1, 3)$ SIC. It is known that N(10, 1, 3) = 10, consequently $wt(x) \ge 10$.

Lemma 8 There is no (35, 11, 2, 3) superimposed code with a column of weight 10, 11 or 12.

Proof Suppose C is a (35, 11, 2, 3) SIC and x is a column of weight v = 10,11 or 12. Then the residual code Res(C, x = 1) is a (v, 10, 1, 3) SIC.

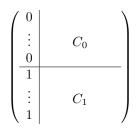
The identity matrix is the unique (10, 10, 1, 3) SIC.

We classified (up to equivalence) all (11, 10, 1, 3) and (12, 10, 1, 3) SIC by a computer program. It turned out that each of the constructed codes contains as a submatrix the identity matrix of order 10.

If we delete the column x and all the rows with weight less or equal to 2 in the code C we obtain an (N', 10, 2, 3) SIC with $N' \leq 25$. This is a contradiction because N(10, 2, 3) = 30.

Lemma 9 There is no (35, 11, 2, 3) superimposed code with a column of weight 15.

Proof By Lemma 8 the possible weights of a column in a (35, 11, 2, 3) SIC are 13, 14 and 15. Suppose C is a (35, 11, 2, 3) SIC and x is a column of weight 15. Then the residual code Res(C, x = 0) is a (20, 10, 2, 2) SIC. There is unique (20, 10, 2, 2) SIC [1]. So the matrix of the code C is of the form:



where C_0 is the unique (20, 10, 2, 2) SIC or its complementary code.

Using an exhaustive computer search we tried to construct the matrix C_1 . It turned out that the extension is impossible.

Lemma 10 There is no (35, 11, 2, 3) superimposed code with a column of weight 14.

Proof It follows from Lemmas 8 and 9 that the possible weights of a column in a (35, 11, 2, 3) SIC are 13 and 14. Suppose C is a (35, 11, 2, 3) SIC and x is a column of weight 14. Then the residual code Res(C, x = 0) is a (21, 10, 2, 2) SIC and the structure of the (35, 11, 2, 3) code matrix is similar to that in Lemma 9. We classified up to equivalence all (21, 10, 2, 2) SIC and obtained that there are 72 possibilities for the matrix C_0 . For each of these cases we tried to construct C_1 -part. It turned out, however, that there is no solution.

Lemma 11 There is no (35, 11, 2, 3) superimposed code with a column of weight 13.

Proof Suppose C is a (35, 11, 2, 3) SIC. It follows from Lemmas 8, 9 and 10 that all columns of C are of weight 13.

Up to equivalence we may assume that the code C has the form similar to that of Lemma 9 where the matrix C_0 is a (22, 10, 2, 2) SIC, and the matrix C_1 is a (13, 10, 1, 3) SIC. Not every (13, 10, 1, 3) SIC could be places at the C_1 -part. The C_1 SIC must have at most 5 row of weight less than 2. Else we could take

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the matrix $\binom{C_0}{C_1}$ and delete the rows of weight less than 2. The remaining matrix would be an (N', 10, 2, 3) SIC with N' < 30 – a contradiction. We obtained that there are exactly 7 inequivalent possibilities for the matrix C_1 . The weight distributions of the columns are the following $(B_i$ is the number of columns of weight i):

C_1	B_1	B_2	B_3	$\frac{B_4}{10}$
1, 2, 3, 4				10
5			1	9
6		1		9
7	1			9

Consequently there are exactly 4 possibilities for the weight distribution of the code C_0 . Using an exhaustive computer search we found 6 inequivalent (22, 10, 2, 2) codes with these weight distributions:

C_0	B_{12}	B_{11}	B_{10}	B_9
1, 2, 3, 4				10
5			1	9
6		1		9

Then we tried to assemble the two parts, (22, 10, 2, 2) and (13, 10, 1, 3) superimposed codes, in such a way that the whole matrix to be a (35, 11, 2, 3) SIC. It turned out, however, that there is no solution.

In this way we proved the main result of this section:

Theorem 12 There is no (35, 11, 2, 3) superimposed code. **Corollary 13** $N(11, 2, 3) \ge 36$.

References

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