Sixth International Workshop on Optimal Codes and Related Topics
June 16-22, 2009, Varna, Bulgaria *pp. 81-86*

Regular union and cover free families

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Abstract. A family of subsets $\mathcal{F}$ of an n-set is r-union-free, if the unions of at most r-tuples of the elements in $\mathcal{F}$ are all different. $\mathcal{F} \subseteq 2^{[n]}$ is r-cover-free, if no set in $\mathcal{F}$ is covered by the union r others. In this paper we give new bounds on the maximum size of these families in the regular case. In coding theoretical setting union-free families and cover-free families correspond to superimposed designs and codes. Since the nature of this problem is rather combinatorial we follow the combinatorial notation.

1 Introduction

Union free families were introduced by Kautz and Singleton [16]. They studied binary codes with the property that the disjunctions (bitwise ORs) of distinct at most r-tuples of codewords are all different. In information theory usually these codes are called 'superimposed' and they have been investigated in several papers on multiple access communication (see, e.g. [6, 15]). Alon and Asodi [1, 2], and De Bonis and Vaccaro [5] studied this problem in a more general setup.

The same problem has been posed – in different terms – by Erdős, Frankl and Füredi ([8], [9]) in combinatorics and by Sós [23] in combinatorial number theory. One can find an easy proof of the best known upper bound of these codes in the papers by Füredi [12] and Ruszinkó [22]. We also have to mention here that this problem has a direct geometry application: a union-free family defines a set of points of exponential size in $\mathbf{R}^n$ such that arbitrary three of them span a triangle with all angles sharp.

A $\mathcal{F} \subseteq 2^{[n]}$ of size t is a union-free family with parameters (n, t, r) $(UF(n, t, r))$ if for arbitrary two distinct subsets $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{F}$ with $0 \leq |\mathcal{A}|, |\mathcal{B}| \leq r$

$$\bigcup_{A \in \mathcal{A}} A \neq \bigcup_{A \in \mathcal{B}} B.$$

A $\mathcal{F} \subseteq 2^{[n]}$ of size t is a cover-free family with parameters (n, t, r) $(CF(n, t, r))$ if for arbitrary $A_0, A_1, A_2, \ldots, A_r \in \mathcal{F}$

$$A_0 \not\subseteq \bigcup_{i=1}^{r} A_i.$$

One can easily see that if $\mathcal{F}$ is $CF(n,t,r)$ then it is $UF(n,t,r)$, and if $\mathcal{F}$ is $UF(n,t,r)$ then it is $UF(n,t,r-1)$. The degree of an element $x \in \{1, \ldots, n\} = [n]$ is the number of members in $\mathcal{F}$ containing x. A family $UF(n,t,r)$ $(CF(n,t,r))$ is $UF(n,t,r,k)$ $(CF(n,t,r,k)$, resp.) if the maximum degree is k.

Quite recently, D'yachkov and Rykov [7] introduced the concept of what they called *optimal* superimposed codes and designs (CF, UF, resp). They observed [7] the following two Propositions.

Proposition 1 *(D'yachokov, Rykov [7]) For an arbitrary $CF(n,t,r-1,k)$ (and thus for an arbitrary $UF(n,t,r,k)$) with $t > k > r \geq 2$, $n \geq \lceil rt/k \rceil$ holds.*

A $CF(n,t,r-1,k)$ or $UF(n,t,r,k)$ is called *optimal* in [7] iff in Proposition 1 equality holds. Although equality only in a very special range of parameters of a $CF(n,t,r-1,k)$ or $UF(n,t,r,k)$ may hold. Thus to avoid any confusion we will call these **regular** ones.

Proposition 2 *(D'yachokov, Rykov [7]) In an arbitrary regular $CF(n,t,r-1,k)$ or $UF(n,t,r,k)$*

- *The size of every set is r (r-uniform);*
- *The degree of every element is k (k-regular);*
- *The maximum pairwise intersection is 1 (1-intersecting).*

Dy'achkov and Rykov [7] considered the case when r divides n (i.e. $n = rq$, and so $t = kq$) and obtained several sufficient conditions for the existence of a regular UF, CF, resp. They investigated the maximum k which still guarantees the existence of a regular $UF(rq,kq,r,k)$ or $CF(rq,kq,r,k)$. On the other hand, equivalently, but following the traditions in extremal set or coding theory, we consider the maximum size $t = t(r,n)$ ($t' = t'(r,n)$ for which a regular $UF(n,t,r,k)$ $(UF(n,t,r,k))$ still exists assuming $n = rq$, and $t = kq$. (Since t and k are proportional, the two approaches are equivalent.) In order to have a clear comparison with our results we will restate the older ones in our terms. Since regular union-free families rise in our opinion more interesting questions then cover free ones, the main part of this paper deals with bounds on $t = t(r,n)$. log stands for the logarithm in base 2.

2 New bounds in the graph case

Dy'achkov and Rykov [7] showed the following

Theorem 1 *Let n be an even integer. Then*

1. *for $t \le n^2/4$ (i.e., $k \le n/2$) there exists a $CF(n,t,1,k)$;*
2. *for $t \le n(\log(n+2)-2)/2$ (i.e., $k \le \log(n+2)-2$) there exists a $UF(n,t,2,k)$.*

This theorem can be sharpened as follows.

Proposition 3 *If n is a positive integer then*

1. *for $t \le n(n-1)/2 = \binom{n}{2}$ (i.e., $k \le n-1$) there exists a $CF(n,t,1,k)$;*
2. *for $t \le \frac{n}{2}\sqrt{\frac{n}{2}} + o\left(n^{3/2}\right)$ (i.e., $k \le \sqrt{\frac{n}{2}} + o\left(\sqrt{n}\right)$) there exists a $UF(n,t,2,k)$.*

Proof (1) It is well-known (and obvious), that for n is even, and $n > k$ there exists a k-regular graph on n vertices. Indeed, by Baranyai's Theorem [4] the edge set of K_n can be decomposed into $n-1$ perfect matchings. Take k of them. For odd n (and k even) by Tutte's theorem K_n can be decomposed into $(n-1)/2$ 2-factors. Take $k/2$ of them. □

(2) For a fixed k consider the following bipartite graph $G=(A,B;E)$. Let $q=q(k)$ be the smallest integer such that $\{0,1,\dots,q-1\}$ contains a Sidon set size k, i.e., a set of non-negative integers $X \subseteq A$ such that for arbitrary quadruple $i,j,m,l \in X$,

$$i + j \not\equiv m + l \pmod{q}.$$

Erdős and Turán [11] showed that $q(k) = k^2 + o(k^2)$. (For more details on this topic see, e.g, the excellent paper of Babai and Sós [3]). Let $A = B = \{0,1,\dots,q-1\}$ and take a Sidon set X in A of size k. Define

$$E = \{(a, a + x \pmod{q}) : \ a \in A,\ x \in X\}.$$

Obviously, a graph G is a $UF(2q,kq,2,k)$ iff it contains no C_3 or C_4. G is bipartite so it does not contain any triangle. We will show that G contains no C_4 either, i.e. it is $UF(2q,kq,2,k)$. Indeed, assume to the contrary that there is a $C_4=(a_1,b_1,a_2,b_2)$, where $a_i \in A$ and $b_i \in B$. Then, by the construction,

$$\begin{aligned} b_1 &\equiv a_1 + i \equiv a_2 + l \pmod{q} \\ b_2 &\equiv a_1 + m \equiv a_2 + j \pmod{q}, \end{aligned}$$

from which $i+j \equiv m+l \pmod q$ follows, a contradiction. By the choice of the parameters, $n = 2q$, $k = \sqrt{\frac{n}{2}} + o(\sqrt{n})$, $t = \frac{n}{2}\sqrt{\frac{n}{2}} + o\left(n^{3/2}\right)$, as claimed. $\square$

For a family of graphs $\mathcal{G}$ let $ex(n;\mathcal{G})$ be the maximum number of edges in a graph with n vertices containing no copy of a graph in $\mathcal{G}$. By theorems Kővári, T. Sós, Turán [17] and Reiman [21] $ex(n;\{C_4\}) \sim n^{3/2}/2$, i.e., our construction in magnitude is tight. To determine the right constant here seems to be a very difficult open question. By Erdős and Simonovits [10] $ex(n;\{C_4,C_5\}) = ex(n;\{C_4,C_3,C_5,\dots C_{2k+1}\dots\}) \sim n^{3/2}/(2\sqrt{2})$, i.e., forbidding C_5 in magnitude is the same a forbidding all non-bipartite graphs. But this is not clear for triangles, it follows that $n^{3/2}/(2\sqrt{2}) \le ex(n;\{C_4,C_3\}) \le n^{3/2}/2$, but the right constant is not known.

3 New bounds in the hypergraph case

For $r \ge 3$ Dyachkov and Rykov [7] showed that

Theorem 2 *For $r \ge 3$ if*

$$t \le \left(\frac{n}{r}\right)^{r/(r-1)} \le \frac{c}{r} n^{1+1/(r-1)} \quad \left(i.e.,\ k \le \left(\frac{n}{r}\right)^{1/(r-1)}\right) \tag{1}$$

then there exists a $UF(n,t,r,k)$.

(Here to make the bound more transparent we bound $r^{-1/(r-1)}$ by an absolute constant c.) In our main theorem we improve this as follows.

Theorem 3 *1. ($r = 3$) For $t \le cn^{3/2}$ (i.e., $k \le cn^{1/2}$) there exists a $UF(n,t,3,k)$;*

2. ($r > 3$) For $t \le n^2 e^{-\beta_r\sqrt{\log n}}$ (i.e., $k \le ne^{-\beta_r\sqrt{\log n}}$) there exists a $UF(n,t,r,k)$.

Notice that for $r = 3$ our bound matches the one of Dyachkov and Rykov. On the other hand, for larger values of r our bound is much stronger: it is almost quadratic in n, while the one in 1 is close to linear. Also notice that a quadratic bound would be the best possible. Indeed, by Proposition 2 a regular union-free family is 1-intersecting. So the sets $A_1, A_2, \dots, A_k$ containing a given element x all disjoint outside $\{x\}$. Thus $n \ge |\cup A_i| = 1 + (r-1)k$. In the quite complicated proof of Theorem 3 we use the famous construction of Behrend of large sets of integers with no three term arithmetic progression.

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