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Good (n, r)-arcs in PG(2, 23)[1]

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Abstract. An (n, r)-arc is a set of n points of a projective plane such that some r, but no $r + 1$ of them, are collinear. The maximum size of an (n, r)-arc in PG(2, q) is denoted by $m_r(2, q)$. In this paper we establish that $m_3(2, 23) \geq 35$, $m_4(2, 23) \geq 58$, $m_5(2, 23) \geq 77$, $m_6(2, 23) \geq 97$ and $m_7(2, 23) \geq 119$.

1 Introduction

Let GF(q) denote the Galois field of q elements and V(3, q) be the vector space of row vectors of length three with entries in GF(q). Let PG(2, q) be the corresponding projective plane. The points of PG(2, q) are the non-zero vectors of V(3, q) with the rule that $X = (x_1, x_2, x_3)$ and $Y = (\lambda x_1, \lambda x_2, \lambda x_3)$ are the same point, where $\lambda \in \mathrm{GF(q)}\setminus\{0\}$. Since any non-zero vector has precisely $q-1$ non-zero scalar multiples, the number of points of PG(2, q) is $\frac{q^3-1}{q-1} = q^2+q+1$.

If the point $P(X)$ is the equivalence class of the vector X, then we will say that X is a vector *representing* $P(X)$. A subspace of dimension one is a set of points all of whose representing vectors form a subspace of dimension two of V(3, q). Such subspaces are called *lines*. The number of lines in PG(2, q) is $q^2 + q + 1$. There are $q + 1$ points on every line and $q + 1$ lines through every point.

Definition 1.1 *An (n, r)-arc is a set of n points of a projective plane such that some r, but no $r + 1$ of them, are collinear.*

Definition 1.2 *Let M be a set of points in any plane. An i-secant is a line meeting M in exactly i points. Define τ_i as the number of i-secants to a set M.*

In terms of τ_i the definition of (n, r)-arc becomes

Definition 1.3 *An (n, r)-arc is a set of n points of a projective plane for which $\tau_i \geq 0$ for $i < r$, $\tau_r > 0$ and $\tau_i = 0$ when $i > r$.*

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In 1947 Bose [4] proved that $m_2(2,q) = q+1$ for q odd, and $m_2(2,q) = q+2$ for q even. From the results of Barlotti [3] and Ball [1] it follows that if $r = (q+1)/2$ or $r = (q+3)/2$ and q is odd prime, then $m_r(2,q) = (r-1)q+1$. So in PG(2, 23) we have:

$$m_2(2,23) = 24, \qquad m_{12}(2,23) = 254, \qquad m_{13}(2,23) = 277.$$

For example, the set of points

$(0,1,0)$	$(1,1,2)$	$(1,1,3)$	$(1,1,4)$	$(1,1,6)$	$(1,1,8)$	$(1,1,b)$	$(1,1,c)$
$(1,1,f)$	$(1,1,h)$	$(1,1,k)$	$(1,1,l)$	$(1,1,m)$	$(1,2,3)$	$(1,2,6)$	$(1,2,7)$
$(1,2,8)$	$(1,2,9)$	$(1,2,a)$	$(1,2,d)$	$(1,2,e)$	$(1,2,f)$	$(1,2,g)$	$(1,2,h)$
$(1,2,l)$	$(1,3,2)$	$(1,3,4)$	$(1,3,5)$	$(1,3,8)$	$(1,3,9)$	$(1,3,a)$	$(1,3,d)$
$(1,3,e)$	$(1,3,f)$	$(1,3,i)$	$(1,3,k)$	$(1,3,m)$	$(1,4,1)$	$(1,4,4)$	$(1,4,6)$
$(1,4,7)$	$(1,4,8)$	$(1,4,b)$	$(1,4,c)$	$(1,4,f)$	$(1,4,g)$	$(1,4,h)$	$(1,4,k)$
$(1,4,n)$	$(1,5,0)$	$(1,5,3)$	$(1,5,4)$	$(1,5,5)$	$(1,5,9)$	$(1,5,b)$	$(1,5,c)$
$(1,5,e)$	$(1,5,i)$	$(1,5,k)$	$(1,5,l)$	$(1,6,1)$	$(1,6,2)$	$(1,6,3)$	$(1,6,4)$
$(1,6,6)$	$(1,6,a)$	$(1,6,d)$	$(1,6,h)$	$(1,6,k)$	$(1,6,l)$	$(1,6,m)$	$(1,6,n)$
$(1,7,0)$	$(1,7,2)$	$(1,7,6)$	$(1,7,7)$	$(1,7,9)$	$(1,7,a)$	$(1,7,d)$	$(1,7,e)$
$(1,7,g)$	$(1,7,h)$	$(1,7,m)$	$(1,8,3)$	$(1,8,5)$	$(1,8,6)$	$(1,8,7)$	$(1,8,9)$
$(1,8,b)$	$(1,8,c)$	$(1,8,e)$	$(1,8,g)$	$(1,8,h)$	$(1,8,i)$	$(1,8,l)$	$(1,9,1)$
$(1,9,5)$	$(1,9,6)$	$(1,9,9)$	$(1,9,a)$	$(1,9,b)$	$(1,9,c)$	$(1,9,d)$	$(1,9,e)$
$(1,9,h)$	$(1,9,i)$	$(1,9,n)$	$(1,a,0)$	$(1,a,1)$	$(1,a,2)$	$(1,a,3)$	$(1,a,8)$
$(1,a,9)$	$(1,a,e)$	$(1,a,f)$	$(1,a,l)$	$(1,a,m)$	$(1,a,n)$	$(1,b,0)$	$(1,b,2)$
$(1,b,3)$	$(1,b,7)$	$(1,b,a)$	$(1,b,b)$	$(1,b,c)$	$(1,b,d)$	$(1,b,g)$	$(1,b,l)$
$(1,b,m)$	$(1,c,3)$	$(1,c,4)$	$(1,c,5)$	$(1,c,7)$	$(1,c,8)$	$(1,c,a)$	$(1,c,d)$
$(1,c,f)$	$(1,c,g)$	$(1,c,i)$	$(1,c,k)$	$(1,c,l)$	$(1,d,1)$	$(1,d,2)$	$(1,d,3)$
$(1,d,5)$	$(1,d,a)$	$(1,d,b)$	$(1,d,c)$	$(1,d,d)$	$(1,d,i)$	$(1,d,l)$	$(1,d,m)$
$(1,d,n)$	$(1,e,0)$	$(1,e,1)$	$(1,e,4)$	$(1,e,7)$	$(1,e,a)$	$(1,e,b)$	$(1,e,c)$
$(1,e,d)$	$(1,e,g)$	$(1,e,k)$	$(1,e,n)$	$(1,f,0)$	$(1,f,2)$	$(1,f,5)$	$(1,f,6)$
$(1,f,8)$	$(1,f,b)$	$(1,f,c)$	$(1,f,f)$	$(1,f,h)$	$(1,f,i)$	$(1,f,m)$	$(1,g,1)$
$(1,g,2)$	$(1,g,7)$	$(1,g,8)$	$(1,g,9)$	$(1,g,b)$	$(1,g,c)$	$(1,g,e)$	$(1,g,f)$
$(1,g,g)$	$(1,g,m)$	$(1,g,n)$	$(1,h,0)$	$(1,h,2)$	$(1,h,4)$	$(1,h,5)$	$(1,h,6)$
$(1,h,7)$	$(1,h,g)$	$(1,h,h)$	$(1,h,i)$	$(1,h,k)$	$(1,h,m)$	$(1,i,1)$	$(1,i,2)$
$(1,i,4)$	$(1,i,5)$	$(1,i,7)$	$(1,i,9)$	$(1,i,e)$	$(1,i,g)$	$(1,i,i)$	$(1,i,k)$
$(1,i,m)$	$(1,i,n)$	$(1,k,0)$	$(1,k,1)$	$(1,k,3)$	$(1,k,5)$	$(1,k,7)$	$(1,k,8)$
$(1,k,f)$	$(1,k,g)$	$(1,k,i)$	$(1,k,l)$	$(1,k,n)$	$(1,l,0)$	$(1,l,1)$	$(1,l,5)$
$(1,l,6)$	$(1,l,8)$	$(1,l,a)$	$(1,l,d)$	$(1,l,f)$	$(1,l,h)$	$(1,l,i)$	$(1,l,n)$
$(1,m,0)$	$(1,m,1)$	$(1,m,3)$	$(1,m,4)$	$(1,m,6)$	$(1,m,9)$	$(1,m,e)$	$(1,m,h)$
$(1,m,k)$	$(1,m,l)$	$(1,m,n)$	$(1,n,0)$	$(1,n,4)$	$(1,n,8)$	$(1,n,9)$	$(1,n,a)$
$(1,n,b)$	$(1,n,c)$	$(1,n,d)$	$(1,n,e)$	$(1,n,f)$	$(1,n,k)$		

forms a $(254, 12)$-arc in PG(2, 23) with secant distribution

$$\tau_0 = 23, \quad \tau_1 = 1, \quad \tau_{11} = 253, \quad \tau_{12} = 276.$$

The elements of GF(23) are denoted by $0, 1, 2, \ldots, 9$, $10 = a$, $11 = b$, $12 = c$, $13 = d$, $14 = e$, $15 = f$, $16 = g$, $17 = h$, $18 = i$, $19 = k$, $20 = l$, $21 = m$, $22 = n$.

In [1] the next theorem is proved:

Theorem 1.1 *Let K be an (n, r)-arc in* PG(2, q) *where q is prime.*

1. *If $r \le (q+1)/2$ then $m_r(2, q) \le (r-1)q + 1$.*

2. *If $r \ge (q+3)/2$ then $m_r(2, q) \le (r-1)q + r - (q+1)/2$.*

It follows from this theorem that

$$m_3(2, 23) \le 47, \quad m_4(2, 23) \le 70,$$

$$m_5(2, 23) \le 93, \quad m_6(2, 23) \le 116, \quad m_7(2, 23) \le 139.$$

A survey of (n, r)-arcs with the best known results was presented in [9]. In the years 2004-2005 many improvements were obtained in [6], [7] and [5]. Summarizing these results, Ball and Hirschfeld presented in [2] a new table with bounds on $m_r(2, q)$ for $q \le 19$. This table can also be found at the website of S. Ball.

2 Arcs in PG(2, 23)

Theorem 2.1 *There exist a $(35, 3)$-arc, a $(58, 4)$-arc, a $(77, 5)$-arc, a $(97, 6)$-arc and a $(119, 7)$-arc in* PG(2,23). *Therefore,*

$$35 \le m_3(2, 23) \le 47, \quad 58 \le m_4(2, 23) \le 70,$$

$$77 \le m_5(2, 23) \le 93, \quad 97 \le m_6(2, 23) \le 116, \quad 119 \le m_7(2, 23) \le 139$$

Proof:
1. The set of points $\mathcal{K}_1$

$$\begin{array}{ccccccc}
(0,1,0), & (0,1,1), & (1,0,0), & (1,0,1), & (1,1,2), & (1,3,h), & (1,4,l), \\
(1,5,8), & (1,5,f), & (1,6,5), & (1,7,4), & (1,7,k), & (1,8,d), & (1,8,m), \\
(1,9,8), & (1,a,6), & (1,a,h), & (1,a,k), & (1,b,9), & (1,b,e), & (1,e,3), \\
(1,e,l), & (1,f,a), & (1,f,d), & (1,h,b), & (1,h,c), & (1,h,n), & (1,k,2), \\
(1,k,m), & (1,l,7), & (1,l,g), & (1,m,5), & (1,m,i), & (1,n,1), & (1,n,n),
\end{array}$$

forms a $(35,3)$-arc in PG(2, 23) with secant distribution

$$\tau_0 = 171, \quad \tau_1 = 61, \quad \tau_2 = 184, \quad \tau_3 = 137.$$

2. Deleting from $\mathcal{K}_1$ the points $(1,1,2)$, $(1,a,k)$ and adding the 25 points

$(0,0,1)$, $(0,1,i)$, $(1,0,9)$, $(1,1,e)$, $(1,2,2)$, $(1,2,k)$, $(1,3,g)$, $(1,3,n)$
$(1,4,2)$, $(1,4,5)$, $(1,5,a)$, $(1,6,9)$, $(1,7,i)$, $(1,8,f)$, $(1,9,0)$, $(1,9,4)$
$(1,b,h)$, $(1,c,b)$, $(1,d,6)$, $(1,d,e)$, $(1,d,f)$, $(1,k,a)$, $(1,l,c)$, $(1,m,d)$
$(1,n,g)$,

we obtain a $(58,4)$-arc in PG(2, 23) with secant distribution

$$\tau_0 = 99, \quad \tau_1 = 44, \quad \tau_2 = 69, \quad \tau_3 = 154, \quad \tau_4 = 187.$$

3. The set of points

$(0,0,1)$, $(0,1,1)$, $(0,1,i)$, $(1,0,0)$, $(1,0,1)$, $(1,0,9)$, $(1,1,e)$, $(1,2,2)$,
$(1,2,6)$, $(1,2,8)$, $(1,2,k)$, $(1,3,h)$, $(1,3,n)$, $(1,4,2)$, $(1,4,5)$, $(1,4,l)$,
$(1,5,8)$, $(1,5,a)$, $(1,5,f)$, $(1,6,5)$, $(1,6,9)$, $(1,6,h)$, $(1,6,n)$, $(1,7,2)$,
$(1,7,a)$, $(1,7,i)$, $(1,7,k)$, $(1,8,d)$, $(1,8,f)$, $(1,8,m)$, $(1,9,0)$, $(1,9,4)$,
$(1,9,8)$, $(1,9,m)$, $(1,a,6)$, $(1,a,e)$, $(1,a,h)$, $(1,a,k)$, $(1,b,3)$, $(1,b,9)$,
$(1,b,e)$, $(1,b,h)$, $(1,c,1)$, $(1,c,b)$, $(1,c,d)$, $(1,d,6)$, $(1,d,e)$, $(1,d,f)$,
$(1,e,3)$, $(1,e,l)$, $(1,f,1)$, $(1,f,a)$, $(1,f,b)$, $(1,f,d)$, $(1,g,3)$, $(1,h,b)$,
$(1,h,h)$, $(1,h,n)$, $(1,i,2)$, $(1,i,6)$, $(1,i,g)$, $(1,i,n)$, $(1,k,2)$, $(1,k,5)$,
$(1,k,a)$, $(1,k,m)$, $(1,l,7)$, $(1,l,c)$, $(1,l,g)$, $(1,m,0)$, $(1,m,5)$, $(1,m,d)$,
$(1,m,i)$, $(1,n,1)$, $(1,n,e)$, $(1,n,g)$, $(1,n,n)$,

forms a $(77,5)$-arc in PG(2, 23) with secant distribution

$$\tau_0 = 53, \quad \tau_1 = 52, \quad \tau_2 = 47, \quad \tau_3 = 81, \quad \tau_4 = 141, \quad \tau_5 = 179.$$

4. The set of points

$(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(0,1,i)$, $(1,0,0)$, $(1,0,1)$, $(1,0,4)$, $(1,0,d)$,
$(1,0,l)$, $(1,1,0)$, $(1,1,2)$, $(1,1,e)$, $(1,1,h)$, $(1,1,i)$, $(1,2,2)$, $(1,2,8)$,
$(1,2,k)$, $(1,3,g)$, $(1,3,h)$, $(1,3,l)$, $(1,3,n)$, $(1,4,1)$, $(1,4,2)$, $(1,4,5)$,
$(1,4,9)$, $(1,4,l)$, $(1,5,8)$, $(1,5,9)$, $(1,5,a)$, $(1,5,e)$, $(1,5,f)$, $(1,6,5)$,
$(1,6,a)$, $(1,7,2)$, $(1,7,4)$, $(1,7,i)$, $(1,7,k)$, $(1,7,m)$, $(1,8,3)$, $(1,8,7)$,
$(1,8,d)$, $(1,8,f)$, $(1,8,m)$, $(1,9,0)$, $(1,9,4)$, $(1,9,8)$, $(1,9,m)$, $(1,a,6)$,
$(1,a,h)$, $(1,a,k)$, $(1,a,n)$, $(1,b,7)$, $(1,b,9)$, $(1,b,e)$, $(1,b,h)$, $(1,b,k)$,
$(1,c,0)$, $(1,c,1)$, $(1,c,b)$, $(1,c,d)$, $(1,d,3)$, $(1,d,6)$, $(1,d,e)$, $(1,d,f)$,
$(1,e,3)$, $(1,e,7)$, $(1,e,l)$, $(1,f,1)$, $(1,f,a)$, $(1,f,b)$, $(1,f,d)$, $(1,g,3)$,
$(1,g,4)$, $(1,g,7)$, $(1,h,3)$, $(1,h,b)$, $(1,h,c)$, $(1,h,h)$, $(1,h,n)$, $(1,i,6)$,
$(1,i,l)$, $(1,i,n)$, $(1,k,2)$, $(1,k,a)$, $(1,k,m)$, $(1,l,5)$, $(1,l,6)$, $(1,l,7)$,
$(1,l,c)$, $(1,l,g)$, $(1,m,0)$, $(1,m,5)$, $(1,m,d)$, $(1,m,i)$, $(1,n,1)$, $(1,n,g)$,
$(1,n,n)$,

forms a $(97, 7)$-arc in PG(2, 23) with secant distribution

$$\tau_0 = 40, \quad \tau_1 = 32, \quad \tau_2 = 28, \quad \tau_3 = 42, \quad \tau_4 = 97, \quad \tau_5 = 158, \quad \tau_6 = 156.$$

5. The set of points

$(1,1,3)$	$(1,1,4)$	$(1,1,8)$	$(1,1,b)$	$(1,1,h)$	$(1,1,l)$	$(1,2,3)$	$(1,2,6)$
$(1,2,7)$	$(1,2,9)$	$(1,2,d)$	$(1,2,g)$	$(1,4,2)$	$(1,4,6)$	$(1,4,8)$	$(1,4,b)$
$(1,4,c)$	$(1,4,g)$	$(1,4,k)$	$(1,5,3)$	$(1,5,5)$	$(1,5,9)$	$(1,5,c)$	$(1,5,e)$
$(1,5,i)$	$(1,6,3)$	$(1,6,6)$	$(1,6,l)$	$(1,7,2)$	$(1,7,6)$	$(1,7,d)$	$(1,7,e)$
$(1,7,g)$	$(1,7,h)$	$(1,7,m)$	$(1,8,9)$	$(1,8,e)$	$(1,8,g)$	$(1,8,h)$	$(1,8,i)$
$(1,8,l)$	$(1,9,1)$	$(1,9,6)$	$(1,9,9)$	$(1,9,d)$	$(1,9,e)$	$(1,9,i)$	$(1,9,n)$
$(1,a,0)$	$(1,a,1)$	$(1,a,2)$	$(1,a,8)$	$(1,a,e)$	$(1,a,f)$	$(1,a,m)$	$(1,b,2)$
$(1,b,3)$	$(1,b,9)$	$(1,b,a)$	$(1,b,b)$	$(1,b,l)$	$(1,b,m)$	$(1,c,4)$	$(1,c,f)$
$(1,c,l)$	$(1,d,1)$	$(1,d,3)$	$(1,d,c)$	$(1,d,d)$	$(1,d,i)$	$(1,d,m)$	$(1,d,n)$
$(1,e,1)$	$(1,e,4)$	$(1,e,b)$	$(1,e,c)$	$(1,e,d)$	$(1,e,m)$	$(1,e,n)$	$(1,f,0)$
$(1,f,2)$	$(1,f,8)$	$(1,f,b)$	$(1,f,c)$	$(1,f,m)$	$(1,g,8)$	$(1,h,0)$	$(1,h,6)$
$(1,h,7)$	$(1,h,g)$	$(1,h,h)$	$(1,h,k)$	$(1,i,4)$	$(1,i,7)$	$(1,i,9)$	$(1,i,i)$
$(1,i,m)$	$(1,i,n)$	$(1,k,1)$	$(1,k,3)$	$(1,k,7)$	$(1,k,8)$	$(1,k,g)$	$(1,k,i)$
$(1,k,l)$	$(1,l,0)$	$(1,l,d)$	$(1,l,h)$	$(1,l,i)$	$(1,m,1)$	$(1,m,4)$	$(1,m,h)$
$(1,m,k)$	$(1,m,l)$	$(1,n,4)$	$(1,n,9)$	$(1,n,b)$	$(1,n,c)$	$(1,n,f)$	

forms a $(119, 7)$-arc in PG(2, 23) with secant distribution

$$\tau_0 = 26, \quad \tau_1 = 19, \quad \tau_2 = 22, \quad \tau_3 = 37, \quad \tau_4 = 44, \quad \tau_5 = 93, \quad \tau_6 = 143, \quad \tau_7 = 169.$$

There exist a close relationship between (n, r)-arcs in PG(2, q) and $[n, 3, d]_q$ codes, given in the next theorem.

Theorem 2.2 [8] *There exist a projective $[n, 3, d]_q$ code if and only if there exist an $(n, n-d)$-arc in* PG(2, q).

From Theorems 2.1 and 2.2 we have the following:

Corollary 2.1 *There exist projective codes with parameters:*

$$[35, 3, 32]_{23}, \quad [58, 3, 54]_{23}, \quad [77, 3, 72]_{23}, \quad [97, 3, 91]_{23}, \quad [119, 3, 112]_{23}.$$

The first three codes are Griesmer codes.

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