Good \((n, r)\)-arcs in \( \text{PG}(2, 23) \)\(^1\)

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**Abstract.** An \((n, r)\)-arc is a set of \(n\) points of a projective plane such that some \(r\), but no \(r+1\) of them, are collinear. The maximum size of an \((n, r)\)-arc in \( \text{PG}(2, q) \) is denoted by \(m_r(2, q)\). In this paper we establish that \(m_3(2, 23) \geq 35\), \(m_4(2, 23) \geq 58\), \(m_5(2, 23) \geq 77\), \(m_6(2, 23) \geq 97\) and \(m_7(2, 23) \geq 119\).

1 Introduction

Let \( \text{GF}(q) \) denote the Galois field of \(q\) elements and \( \text{V}(3, q) \) be the vector space of row vectors of length three with entries in \( \text{GF}(q) \). Let \( \text{PG}(2, q) \) be the corresponding projective plane. The points of \( \text{PG}(2, q) \) are the non-zero vectors of \( \text{V}(3, q) \) with the rule that \( X = (x_1, x_2, x_3) \) and \( Y = (\lambda x_1, \lambda x_2, \lambda x_3) \) are the same point, where \( \lambda \in \text{GF}(q) \setminus \{0\} \). Since any non-zero vector has precisely \(q-1\) non-zero scalar multiples, the number of points of \( \text{PG}(2, q) \) is \(q^3 - q = q^2 + q + 1\).

If the point \( P(X) \) is the equivalence class of the vector \( X \), then we will say that \( X \) is a vector representing \( P(X) \). A subspace of dimension one is a set of points all of whose representing vectors form a subspace of dimension two of \( \text{V}(3, q) \). Such subspaces are called lines. The number of lines in \( \text{PG}(2, q) \) is \(q^2 + q + 1\). There are \(q + 1\) points on every line and \(q + 1\) lines through every point.

**Definition 1.1** An \((n, r)\)-arc is a set of \(n\) points of a projective plane such that some \(r\), but no \(r+1\) of them, are collinear.

**Definition 1.2** Let \( M \) be a set of points in any plane. An \(i\)-secant is a line meeting \( M \) in exactly \(i\) points. Define \( \tau_i \) as the number of \(i\)-secants to a set \( M \).

In terms of \(\tau_i\) the definition of \((n, r)\)-arc becomes

**Definition 1.3** An \((n, r)\)-arc is a set of \(n\) points of a projective plane for which \(\tau_i \geq 0\) for \(i < r\), \(\tau_r > 0\) and \(\tau_i = 0\) when \(i > r\).

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In 1947 Bose [4] proved that $m_2(2,q) = q + 1$ for $q$ odd, and $m_2(2,q) = q + 2$ for $q$ even. From the results of Barlotti [3] and Ball [1] it follows that if $r = (q + 1)/2$ or $r = (q + 3)/2$ and $q$ is odd prime, then $m_r(2,q) = (r-1)q + 1$. So in PG(2, 23) we have:

$$m_2(2, 23) = 24, \quad m_{12}(2, 23) = 254, \quad m_{13}(2, 23) = 277.$$ 

For example, the set of points

\[
\begin{align*}
(0, 1, 0) & (1, 1, 2) & (1, 1, 3) & (1, 1, 4) & (1, 1, 5) & (1, 1, 6) & (1, 1, 7) & (1, 1, 8) & (1, 1, 9) & (1, 1, b) & (1, 1, c) \\
(1, 1, f) & (1, 1, h) & (1, 1, k) & (1, 1, l) & (1, 1, m) & (1, 1, n) & (1, 2, 3) & (1, 2, 6) & (1, 2, 7) & (1, 2, 9) & (1, 2, a) & (1, 2, b) & (1, 2, c) & (1, 2, d) & (1, 2, e) & (1, 2, f) & (1, 2, g) & (1, 2, h) \\
(1, 2, l) & (1, 3, 2) & (1, 3, 4) & (1, 3, 5) & (1, 3, 8) & (1, 3, 9) & (1, 3, a) & (1, 3, d) & (1, 3, e) & (1, 3, f) & (1, 3, g) & (1, 3, h) & (1, 3, i) & (1, 3, k) & (1, 3, m) & (1, 4, 1) & (1, 4, 4) & (1, 4, 6) & (1, 4, 7) & (1, 4, 8) & (1, 4, b) & (1, 4, c) & (1, 4, f) & (1, 4, g) & (1, 4, h) & (1, 4, k) \\
(1, 4, n) & (1, 5, 0) & (1, 5, 3) & (1, 5, 4) & (1, 5, 5) & (1, 5, 6) & (1, 5, 9) & (1, 5, b) & (1, 5, c) & (1, 5, e) & (1, 5, i) & (1, 5, k) & (1, 5, l) & (1, 6, 1) & (1, 6, 2) & (1, 6, 3) & (1, 6, 4) & (1, 6, 6) & (1, 6, a) & (1, 6, d) & (1, 6, h) & (1, 6, k) & (1, 6, l) & (1, 6, m) & (1, 6, n) \\
(1, 7, 0) & (1, 7, 2) & (1, 7, 6) & (1, 7, 7) & (1, 7, 9) & (1, 7, a) & (1, 7, d) & (1, 7, e) & (1, 7, g) & (1, 7, h) & (1, 7, m) & (1, 8, 3) & (1, 8, 5) & (1, 8, 6) & (1, 8, 7) & (1, 8, 9) & (1, 8, b) & (1, 8, c) & (1, 8, e) & (1, 8, g) & (1, 8, h) & (1, 8, i) & (1, 8, l) & (1, 9, 1) & (1, 9, 5) & (1, 9, 6) & (1, 9, 9) & (1, 9, a) & (1, 9, b) & (1, 9, c) & (1, 9, d) & (1, 9, e) \\
(1, 9, h) & (1, 9, i) & (1, 9, n) & (1, a, 0) & (1, a, 1) & (1, a, 2) & (1, a, 3) & (1, a, 8) & (1, a, 9) & (1, a, c) & (1, a, e) & (1, a, f) & (1, a, l) & (1, a, m) & (1, a, n) & (1, b, 0) & (1, b, 2) & (1, b, 3) & (1, b, 7) & (1, b, a) & (1, b, b) & (1, b, c) & (1, b, d) & (1, b, g) & (1, b, l) \\
(1, b, m) & (1, c, 3) & (1, c, 4) & (1, c, 5) & (1, c, 7) & (1, c, 8) & (1, c, a) & (1, c, d) & (1, c, f) & (1, c, g) & (1, c, i) & (1, c, k) & (1, c, l) & (1, d, 1) & (1, d, 2) & (1, d, 3) & (1, d, 5) & (1, d, a) & (1, d, b) & (1, d, c) & (1, d, d) & (1, d, i) & (1, d, l) & (1, d, m) \\
(1, d, n) & (1, e, 0) & (1, e, 1) & (1, e, 4) & (1, e, 7) & (1, e, a) & (1, e, b) & (1, e, c) & (1, e, d) & (1, e, g) & (1, e, k) & (1, e, n) & (1, f, 0) & (1, f, 2) & (1, f, 4) & (1, f, 5) & (1, f, 6) & (1, f, 8) & (1, f, b) & (1, f, c) & (1, f, f) & (1, f, h) & (1, f, i) & (1, f, m) & (1, g, 1) \\
(1, g, 2) & (1, g, 7) & (1, g, 8) & (1, g, 9) & (1, g, b) & (1, g, c) & (1, g, e) & (1, g, f) & (1, g, g) & (1, g, m) & (1, g, n) & (1, h, 0) & (1, h, 2) & (1, h, 4) & (1, h, 5) & (1, h, 6) & (1, h, 7) & (1, h, g) & (1, h, h) & (1, h, i) & (1, h, k) & (1, h, m) & (1, i, 1) & (1, i, 2) & (1, i, 4) & (1, i, 5) & (1, i, 7) & (1, i, 9) & (1, i, e) & (1, i, g) & (1, i, i) & (1, i, k) \\
(1, i, m) & (1, i, n) & (1, k, 0) & (1, k, 1) & (1, k, 3) & (1, k, 5) & (1, k, 7) & (1, k, 8) & (1, k, f) & (1, k, g) & (1, k, i) & (1, k, l) & (1, k, m) & (1, l, 0) & (1, l, 1) & (1, l, 5) & (1, l, 6) & (1, l, 8) & (1, l, a) & (1, l, d) & (1, l, f) & (1, l, h) & (1, l, i) & (1, l, n) & (1, m, 0) & (1, m, 1) & (1, m, 3) & (1, m, 4) & (1, m, 6) & (1, m, 9) & (1, m, c) & (1, m, h) \\
(1, m, k) & (1, m, l) & (1, m, n) & (1, n, 0) & (1, n, 4) & (1, n, 8) & (1, n, 9) & (1, n, a) & (1, n, b) & (1, n, c) & (1, n, d) & (1, n, e) & (1, n, f) & (1, n, k) & (1, n, l) & (1, n, m)
forms a \((254, 12)\)-arc in \(PG(2, 23)\) with secant distribution
\[
\tau_0 = 23, \quad \tau_1 = 1, \quad \tau_{11} = 253, \quad \tau_{12} = 276.
\]
The elements of \(GF(23)\) are denoted by \(0, 1, 2, \ldots, 10 = a, 11 = b, 12 = c, 13 = d, 14 = e, 15 = f, 16 = g, 17 = h, 18 = i, 19 = k, 20 = l, 21 = m, 22 = n\).

In [1] the next theorem is proved:

**Theorem 1.1** Let \(K\) be an \((n, r)\)-arc in \(PG(2, q)\) where \(q\) is prime.

1. If \(r \leq (q + 1)/2\) then \(m_r(2, q) \leq (r - 1)q + 1\).
2. If \(r \geq (q + 3)/2\) then \(m_r(2, q) \leq (r - 1)q + r - (q + 1)/2\).

It follows from this theorem that
\[
m_3(2, 23) \leq 47, \quad m_4(2, 23) \leq 70, \quad m_5(2, 23) \leq 93, \quad m_6(2, 23) \leq 116, \quad m_7(2, 23) \leq 139.
\]

A survey of \((n, r)\)-arcs with the best known results was presented in [9]. In the years 2004-2005 many improvements were obtained in [6], [7] and [5]. Summarizing these results, Ball and Hirschfeld presented in [2] a new table with bounds on \(m_r(2, q)\) for \(q \leq 19\). This table can also be found at the website of S. Ball.

## 2 Arcs in \(PG(2, 23)\)

**Theorem 2.1** There exist a \((35, 3)\)-arc, a \((58, 4)\)-arc, a \((77, 5)\)-arc, a \((97, 6)\)-arc and a \((119, 7)\)-arc in \(PG(2, 23)\). Therefore,
\[
35 \leq m_3(2, 23) \leq 47, \quad 58 \leq m_4(2, 23) \leq 70, \quad 77 \leq m_5(2, 23) \leq 93, \quad 97 \leq m_6(2, 23) \leq 116, \quad 119 \leq m_7(2, 23) \leq 139.
\]

**Proof:**
1. The set of points \(K_1\)
\[
\begin{align*}
(0, 1, 0), & \quad (0, 1, 1), \quad (1, 0, 0), \quad (1, 0, 1), \quad (1, 1, 2), \quad (1, 3, h), \quad (1, 4, l), \\
(1, 5, 8), & \quad (1, 5, f), \quad (1, 6, 5), \quad (1, 7, 4), \quad (1, 7, k), \quad (1, 8, d), \quad (1, 8, m), \\
(1, 9, 8), & \quad (1, a, 6), \quad (1, a, h), \quad (1, a, k), \quad (1, b, 9), \quad (1, b, e), \quad (1, e, 3), \\
(1, e, l), & \quad (1, f, a), \quad (1, f, d), \quad (1, h, b), \quad (1, h, c), \quad (1, h, n), \quad (1, k, 2), \\
(1, k, m), & \quad (1, l, 7), \quad (1, l, g), \quad (1, m, 5), \quad (1, m, i), \quad (1, n, 1), \quad (1, n, n),
\end{align*}
\]
forms a \((35, 3)\)-arc in \(\text{PG}(2, 23)\) with secant distribution
\[\tau_0 = 171, \quad \tau_1 = 61, \quad \tau_2 = 184, \quad \tau_3 = 137.\]

2. Deleting from \(K_1\) the points \((1, 1, 2), (1, a, k)\) and adding the 25 points
\[
\begin{align*}
(0, 0, 1), & \quad (0, 1, i), \quad (1, 0, 9), \quad (1, 1, e), \quad (1, 2, 2), \quad (1, 2, k), \quad (1, 3, g), \quad (1, 3, n) \\
(1, 4, 2), & \quad (1, 4, 5), \quad (1, 5, a), \quad (1, 6, 9), \quad (1, 7, i), \quad (1, 8, f), \quad (1, 9, 0), \quad (1, 9, 4) \\
(1, b, h), & \quad (1, c, b), \quad (1, d, 6), \quad (1, d, e), \quad (1, d, f), \quad (1, k, a), \quad (1, l, c), \quad (1, m, d) \\
(1, n, g), & \quad
\end{align*}
\]
we obtain a \((58, 4)\)-arc in \(\text{PG}(2, 23)\) with secant distribution
\[\tau_0 = 99, \quad \tau_1 = 44, \quad \tau_2 = 69, \quad \tau_3 = 154, \quad \tau_4 = 187.\]

3. The set of points
\[
\begin{align*}
(0, 0, 1), & \quad (0, 1, i), \quad (1, 0, 9), \quad (1, 0, 1), \quad (1, 0, 9), \quad (1, 1, e), \quad (1, 2, 2), \\
(1, 2, 6), & \quad (1, 2, 8), \quad (1, 2, k), \quad (1, 3, h), \quad (1, 3, n), \quad (1, 4, 2), \quad (1, 4, 5), \quad (1, 4, l) \\
(1, 5, 8), & \quad (1, 5, a), \quad (1, 5, f), \quad (1, 6, 5), \quad (1, 6, 9), \quad (1, 6, h), \quad (1, 6, n), \quad (1, 7, 2), \\
(1, 7, a), & \quad (1, 7, i), \quad (1, 7, k), \quad (1, 8, d), \quad (1, 8, f), \quad (1, 8, m), \quad (1, 9, 0), \quad (1, 9, 4), \\
(1, 9, 8), & \quad (1, 9, m), \quad (1, a, 6), \quad (1, a, e), \quad (1, a, h), \quad (1, a, k), \quad (1, b, 3), \quad (1, b, 9), \\
(1, b, e), & \quad (1, b, h), \quad (1, c, 1), \quad (1, c, b), \quad (1, c, d), \quad (1, d, 6), \quad (1, d, e), \quad (1, d, f), \\
(1, e, 3), & \quad (1, e, l), \quad (1, f, 1), \quad (1, f, a), \quad (1, f, b), \quad (1, f, d), \quad (1, g, 3), \quad (1, h, b), \\
(1, h, h), & \quad (1, h, n), \quad (1, i, 2), \quad (1, i, 6), \quad (1, i, g), \quad (1, i, n), \quad (1, k, 2), \quad (1, k, 5), \\
(1, k, a), & \quad (1, k, m), \quad (1, l, 7), \quad (1, l, c), \quad (1, l, g), \quad (1, m, 0), \quad (1, m, 5), \quad (1, m, d), \\
(1, m, i), & \quad (1, n, 1), \quad (1, n, e), \quad (1, n, g), \quad (1, n, n),
\end{align*}
\]
forms a \((77, 5)\)-arc in \(\text{PG}(2, 23)\) with secant distribution
\[\tau_0 = 53, \quad \tau_1 = 52, \quad \tau_2 = 47, \quad \tau_3 = 81, \quad \tau_4 = 141, \quad \tau_5 = 179.\]

4. The set of points
\[
\begin{align*}
(0, 0, 1), & \quad (0, 1, 0), \quad (0, 1, i), \quad (1, 0, 0), \quad (1, 0, 1), \quad (1, 0, 4), \quad (1, 0, d), \\
(1, 0, l), & \quad (1, 1, 0), \quad (1, 1, 2), \quad (1, 1, e), \quad (1, 1, h), \quad (1, 1, i), \quad (1, 2, 2), \quad (1, 2, 8), \\
(1, 2, k), & \quad (1, 3, g), \quad (1, 3, h), \quad (1, 3, l), \quad (1, 3, n), \quad (1, 4, 1), \quad (1, 4, 2), \quad (1, 4, 5), \\
(1, 4, 9), & \quad (1, 4, l), \quad (1, 5, 8), \quad (1, 5, 9), \quad (1, 5, a), \quad (1, 5, e), \quad (1, 5, f), \quad (1, 6, 5), \\
(1, 6, a), & \quad (1, 7, 2), \quad (1, 7, 4), \quad (1, 7, i), \quad (1, 7, k), \quad (1, 7, m), \quad (1, 8, 3), \quad (1, 8, 7), \\
(1, 8, d), & \quad (1, 8, f), \quad (1, 8, m), \quad (1, 9, 0), \quad (1, 9, 4), \quad (1, 9, 8), \quad (1, 9, m), \quad (1, a, 6), \\
(1, a, h), & \quad (1, a, k), \quad (1, a, n), \quad (1, b, 7), \quad (1, b, 9), \quad (1, b, e), \quad (1, b, h), \quad (1, b, k), \\
(1, c, 0), & \quad (1, c, 1), \quad (1, c, b), \quad (1, c, d), \quad (1, d, 3), \quad (1, d, 6), \quad (1, d, e), \quad (1, d, f), \\
(1, c, 3), & \quad (1, e, 7), \quad (1, e, l), \quad (1, f, 1), \quad (1, f, a), \quad (1, f, b), \quad (1, f, d), \quad (1, g, 3), \\
(1, g, 4), & \quad (1, g, 7), \quad (1, h, 3), \quad (1, h, b), \quad (1, h, c), \quad (1, h, d), \quad (1, h, n), \quad (1, i, 6), \\
(1, i, l), & \quad (1, i, n), \quad (1, k, 2), \quad (1, k, 4), \quad (1, k, a), \quad (1, k, m), \quad (1, l, 5), \quad (1, l, 6), \quad (1, l, 7), \\
(1, l, c), & \quad (1, l, g), \quad (1, m, 0), \quad (1, m, 5), \quad (1, m, d), \quad (1, m, i), \quad (1, n, 1), \quad (1, n, g), \quad (1, n, n),
\end{align*}
\]
forms a $\tau_0 = 40$, $\tau_1 = 32$, $\tau_2 = 28$, $\tau_3 = 42$, $\tau_4 = 97$, $\tau_5 = 158$, $\tau_6 = 156$.

5. The set of points

$(1,1,3)$ $(1,1,4)$ $(1,1,8)$ $(1,1,b)$ $(1,1,h)$ $(1,1,l)$ $(1,2,3)$ $(1,2,6)$
$(1,2,7)$ $(1,2,9)$ $(1,2,d)$ $(1,2,g)$ $(1,4,2)$ $(1,4,6)$ $(1,4,8)$ $(1,4,b)$
$(1,4,c)$ $(1,4,g)$ $(1,4,k)$ $(1,5,3)$ $(1,5,5)$ $(1,5,9)$ $(1,5,c)$ $(1,5,e)$
$(1,5,i)$ $(1,6,3)$ $(1,6,6)$ $(1,6,l)$ $(1,7,2)$ $(1,7,6)$ $(1,7,d)$ $(1,7,e)$
$(1,7,g)$ $(1,7,h)$ $(1,7,m)$ $(1,8,9)$ $(1,8,e)$ $(1,8,g)$ $(1,8,h)$ $(1,8,i)$
$(1,8,l)$ $(1,9,1)$ $(1,9,6)$ $(1,9,9)$ $(1,9,d)$ $(1,9,c)$ $(1,9,i)$ $(1,9,n)$
$(1,a,0)$ $(1,a,1)$ $(1,a,2)$ $(1,a,8)$ $(1,a,e)$ $(1,a,f)$ $(1,a,m)$ $(1,b,2)$
$(1,b,3)$ $(1,b,9)$ $(1,b,a)$ $(1,b,b)$ $(1,b,l)$ $(1,b,m)$ $(1,c,4)$ $(1,c,f)$
$(1,c,l)$ $(1,d,1)$ $(1,d,3)$ $(1,d,c)$ $(1,d,d)$ $(1,d,i)$ $(1,d,m)$ $(1,d,n)$
$(1,e,1)$ $(1,e,4)$ $(1,e,b)$ $(1,e,c)$ $(1,e,d)$ $(1,e,m)$ $(1,e,n)$ $(1,f,0)$
$(1,f,2)$ $(1,f,8)$ $(1,f,b)$ $(1,f,c)$ $(1,f,m)$ $(1,g,8)$ $(1,h,0)$ $(1,h,6)$
$(1,h,7)$ $(1,h,g)$ $(1,h,h)$ $(1,h,k)$ $(1,i,4)$ $(1,i,7)$ $(1,i,9)$ $(1,i,i)$
$(1,i,m)$ $(1,i,n)$ $(1,k,1)$ $(1,k,3)$ $(1,k,7)$ $(1,k,8)$ $(1,k,g)$ $(1,k,i)$
$(1,k,l)$ $(1,l,0)$ $(1,l,d)$ $(1,l,h)$ $(1,l,i)$ $(1,m,1)$ $(1,m,4)$ $(1,m,h)$
$(1,m,k)$ $(1,m,l)$ $(1,n,4)$ $(1,n,9)$ $(1,n,b)$ $(1,n,c)$ $(1,n,f)$

forms a $(119,7)$-arc in $\text{PG}(2, 23)$ with secant distribution

$\tau_0 = 26$, $\tau_1 = 19$, $\tau_2 = 22$, $\tau_3 = 37$, $\tau_4 = 44$, $\tau_5 = 93$, $\tau_6 = 143$, $\tau_7 = 169$.

There exist a close relationship between $(n,r)$-arcs in $\text{PG}(2, q)$ and $[n, 3, d]_q$ codes, given in the next theorem.

**Theorem 2.2** [8] There exist a projective $[n, 3, d]_q$ code if and only if there exist an $(n, n - d)$-arc in $\text{PG}(2, q)$.

From Theorems 2.1 and 2.2 we have the following:

**Corollary 2.1** There exist projective codes with parameters:


The first three codes are Griesmer codes.

**References**


