CODING THEORY

On the Construction of $q$-ary Equidistant Codes

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Abstract—The problem of constructing equidistant codes over an alphabet of an arbitrary size $q$ is considered. Some combinatorial constructions and computer-based search methods are presented. All maximal equidistant codes with distances 3 and 4 are found.

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1. INTRODUCTION

Let $q$ be a natural number, $q \geq 2$. Denote by $Q$ the set $\{0,1,\ldots,q-1\}$. Let $Q^n$ be the set of all vectors of length $n$ over $Q$. Any subset $C$ of $Q^n$ is called a code and is denoted by $C_q(n,d,N)$, where $n$ is the length of the code, $d$ is the minimum (Hamming) distance of the code, and $N$ is the number of codewords (the cardinality of the code). Such a code is called equidistant and is denoted by $E_q(n,d,N)$ if any two distinct codewords are at the same distance $d$ from each other. Define the number

$$B_q(n,d) = \max\{N : \exists E_q(n,d,N)\},$$

which is the largest possible value of $N$ when the parameters $q$, $n$, and $d$ are fixed. A code $C$ is said to be constant-weight and is denoted by $C_q(n,w,d,N)$ if all codewords have the same weight $w$. We will also consider codes that are both equidistant and constant-weight. Such a code is denoted by $E_q(n,w,d,N)$. Now denote by $B_q(n,w,d)$ the largest possible size $N$ of such a code when the parameters $q$, $n$, $w$, and $d$ are fixed.

One of the main open problem of algebraic coding theory is finding the best possible parameters of equidistant and equidistant constant-weight codes, as well as constructing such codes with the best possible parameters.

Equidistant codes are considered in a large number of papers. In particular, we refer the reader to the papers [1–8]. In [6–11], equidistant constant-weight codes are studied.

In the present paper, the problem of constructing equidistant codes over an alphabet of an arbitrary size $q$ is considered. For this, both combinatorial and computer-based methods are used. To construct equidistant code, we apply computer search in cases where combinatorial construction methods cannot be used. Some known combinatorial constructions and general bounds for equidistant codes are presented in Section 2. Some new combinatorial methods to construct equidistant codes are considered in Section 3. In Section 4, all maximal equidistant codes with distances $d = 3$ and $d = 4$ are enumerated. In Section 5, computer search is considered and a table of maximal $q$-ary equidistant codes for the values $q \leq 9$, $d \leq 10$, and $n \leq 10$ is given.

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2. PRELIMINARY RESULTS

Assume that the alphabet \( Q = \{0,1,\ldots,q-1\} \) has at least the structure of an abelian group (written additively).

**Definition 1.** Let \( D \) be an \( m \times n \) matrix over \( Q \) satisfying the following property: for any two distinct rows \( x = (x_1,\ldots,x_n) \) and \( y = (y_1,\ldots,y_n) \) of \( D \), the row \( x - y = (x_1 - y_1,\ldots,x_n - y_n) \), i.e., their componentwise difference, contains any element of \( Q \) the same number of times, say \( \lambda \), so that \( n = \lambda q \). We say that \( D \) is a difference matrix and denote it by \( D_q(m,n) \) (by \( D_q(n) \) if \( m = n \)).

Such square matrices were introduced in [12] (see also [13, 14]). The following lemma from [12] gives a wide class of such square matrices (for a the simple construction of all such matrices, see [2]).

**Lemma 1** [12]. Let \( q = g^h \) and \( n = g^{h+\ell} \), where \( g = p^s \), \( s = 1,2,\ldots \). Let \( p \) be a prime number and \( h \) and \( \ell \) be arbitrary natural numbers. Let \( Q \) be an \( h \)-dimensional vector space over the finite field \( \mathbb{F}_g \). Then there exists a difference matrix \( D_q(n) \) over \( Q \).

For any \( a \in Q \), denote by \( D_q^{(a)}(n) \) the matrix obtained from \( D_q(n) \) by adding the element \( a \) to all elements of \( D_q(n) \). It is clear that \( D_q^{(a)}(n) \) is also a difference matrix with the same parameters. The following lemma is a direct consequence of the definitions of \( D_q(n) \) and \( D_q^{(a)}(n) \).

**Lemma 2** [2]. For any pair of distinct rows of \( D_q^{(a)}(n) \), the (Hamming) distance between them is \( d = (q - 1)n/q \). For the \( i \)-th row of \( D_q^{(a)}(n) \) and the \( j \)-th row of \( D_q^{(b)}(n) \), \( a \neq b \), we have

\[
d = \begin{cases} 
  n & \text{if } i = j, \\
  (q - 1)n/q & \text{if } i \neq j.
\end{cases}
\]

For an \( E_q(n,d,N) \)-code (respectively, an \( E_q(n,w,d,N) \)-code) \( C \) over an alphabet \( Q \), denote by \( V = V_C \) an \( N \times n \) matrix over \( Q \) formed by all codewords of \( C \) written in a certain order. We say that an equidistant code \( E_q(n,d,N) \) (respectively, a code \( E_q(n,w,d,N) \)) is impasse if it is impossible to add any more codeword without reducing the minimum distance of the code.

**Definition 2.** We say that an equidistant code \( E_q(n,d,N) \) (respectively, a code \( E_q(n,w,d,N) \)) is maximal if equidistant codes \( E_q(n,d,N + 1) \) and \( E_q(n,d + 1,N) \) (respectively, \( E_q(n,w,d,N + 1) \) and \( E_q(n,w,d + 1,N) \)) do not exist. We say that a maximal equidistant code \( E_q(n,d,N) \) (respectively, \( E_q(n,w,d,N) \)) is optimal if every column in the code matrix \( V_C \) contains each (respectively, each nonzero) element of \( Q \) the same number of times, say \( \mu \), so that \( N = q\mu \) (respectively, \( N = (q - 1)(n/w)\mu \)).

It is easy to see that such an optimal \( E_q(n,d,N) \) code has the maximum possible distance \( d \) [1], where

\[
d = n \frac{N(q - 1)}{q(N - 1)}.
\]

Deleting the zero codeword in an \( E_q(n,d,N) \)-code obviously gives an equidistant constant-weight \( E_q(n,w,d,N) \)-code with \( w = d \).

The following simple statement for equidistant codes is of use when considering codes with distances 3 and 4.

**Lemma 3.** Let \( C \) be an arbitrary \( E_q(n,d,N) \)-code over an alphabet of size \( q \), and let \( V_C \) be a matrix formed by all codewords of \( C \). Denote by \( \xi_\gamma^{(j)} \) the multiplicity of occurrence of an element \( \gamma \in Q \) in the \( j \)-th column of \( V_C \). Then for the code \( C \) we have the following equality:

\[
\Gamma_C := \sum_{j=1}^{n} \sum_{\gamma \in Q} \xi_\gamma^{(j)}(\xi_\gamma^{(j)} - 1) = (n - d)N(N - 1).
\]
Proof. The proof follows from the definitions of $\Gamma_C$ and the multiplicity numbers $\xi^{(j)}_\gamma$ and from the equidistance property of the code $C$. △

It is convenient to represent the multiplicities of occurrences $\xi^{(j)}_\gamma$ in the form of an $q \times n$ multiplicity matrix $\Phi_C = (\xi^{(j)}_\gamma)$. For an $E_q(n,d,N)$ code $C$ containing the all-zero codeword, the matrix $\Phi_C$ obviously satisfies (2) and the following two properties:

$$\sum_{\gamma \in Q, \gamma \neq 0} \sum_{j=1}^n \xi^{(j)}_\gamma = d(N - 1)$$

(3)

and

$$\sum_{\gamma \in Q} \xi^{(j)}_\gamma = N, \quad j = 1, \ldots, n.$$  

(4)

For a given vector $v = (v_1, \ldots, v_n)$ in $Q^n$, denote by $\text{supp}(v) = \{ \ell \mid v_\ell \neq 0 \}$ its support. Let $J = \{1, 2, \ldots, n\}$ be the coordinate set of $Q^n$. For a code $C \subset Q^n$, denote by $\text{supp}(C) \subset J$ its support:

$$\text{supp}(C) = \bigcup_{c \in C} \text{supp}(c).$$

We also need the notion of a 2-design (see [13,14]), which is convenient to define in terms of its binary incidence matrix.

Definition 3. Let $C$ be a binary equidistant constant-weight $E_2(n,w,d,N)$-code, and let $V_C$ be the code matrix of $C$. If every column of $V_C$ contains exactly $k$ nonzero elements, then $V_C$ is an incidence matrix of a BIB design with parameters $(v,b,r,k,\lambda)$, where

$$v = N, \quad b = n, \quad r = w, \quad k = Nw/n, \quad \lambda = w - d/2.$$  

A BIB design $(v,b,r,k,\lambda)$ is also called a 2-design $(v,k,\lambda)$, where

$$b = \frac{v(v-1)}{k(k-1)} \lambda \quad \text{and} \quad r = \frac{v-1}{k-1} \lambda.$$  

(5)

Such a 2-design $(v,k,\lambda)$ is said to be symmetric (an SBIB design) if $v = b$, which is equivalent to the condition $(v - 1)\lambda = k(k - 1)$.

Two families of classical symmetric SBIB designs $(v,k,\lambda)$ are, first, a projective plane of order $k$, which is an SBIB design $(k^2 + k + 1, k + 1, 1)$, and second, a Hadamard SBIB design $(4m - 1, 2m - 1, m - 1)$. The latter arises from a Hadamard matrix $H_n$ of order $n = 4m$, i.e., an $n \times n$ matrix with elements $\pm 1$ whose rows are orthogonal over the real field.

In general, the connection between BIB designs and binary equidistant codes can be formulated in the following form, which is a variant of the theorem proved in [6].

Lemma 4 [6]. The existence of a BIB design $(v,b,r,k,\lambda)$ is equivalent to the existence of an optimal binary equidistant constant-weight code $E_2(n,w,d,N)$, where

$$n = b, \quad w = r, \quad d = 2(r - \lambda), \quad N = v.$$  

(6)

Clearly, a symmetric 2-design $(v,k,\lambda)$ is equivalent to a symmetric optimal equidistant code $E_2(v,k,2(k-\lambda),v)$. The two special cases of symmetric 2-designs mentioned above were considered in [3,15]. Here we give a result of [15], which relates Hadamard matrices with equidistant codes.

Lemma 5 [15]. The existence of a Hadamard matrix $H_n$ of order $n$ is equivalent to the existence of an optimal binary equidistant code $E_2(n-1,n/2,n)$. 

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Definition 4. Let $C$ be a binary equidistant constant-weight code $E_2(n, w, d, N)$ which is a 2-design $(v, b, r, k, \lambda)$, i.e.,

$$v = N, \quad b = n, \quad r = w, \quad k = Nw/n, \quad \lambda = w - d/2.$$ 

Assume that the matrix $V_C$ of all codewords of $C$ can be represented in the form

$$V_C = \left( V_1 \mid V_2 \mid \ldots \mid V_r \right),$$

where each row of the submatrix $V_i$, $i = 1, 2, \ldots, r$, has exactly one nonzero position. Then such a 2-design is called resolvable (for short, an RBIB design).

Theorem 1 (trivial values). The numbers $B_q(n, d)$ and $B_q(n, w, d)$ satisfy the following inequalities:

1. $B_q(n, n) = q$;
2. $B_q(n, w, n) \leq q$;
3. $B_q(n, n, d) = B_{q-1}(n, d)$;
4. $B_q(n + 1, w, d) \geq B_q(n, w, d)$;
5. $B_q(n + 1, w + 1, d) \geq B_q(n, w, d)$.

Equidistant codes and equidistant constant-weight codes are closely related, as is shown, in particular, by the following statement.

Theorem 2 [8]. We have

$$B_q(n, d) = 1 + B_q(n, d, d).$$

The classical Plotkin bound is as follows.

Theorem 3 [16]. For any $q$, $n$, and $d$ such that $d > n(q - 1)/q$, we have

$$B_q(n, d) \leq \frac{dq}{dq - n(q - 1)}.$$  \hfill (7)

The exact equality in (7) is equivalent [1] to the existence of an optimal equidistant code $E_q(n, d, B_q(n, d))$ whose distance satisfies (2). Clearly, such a code can be repeated several times, preserving the optimality property.

The general result of Delsarte [17] implies the following statement.

Theorem 4. For any $q$, $n$, and $d$, we have

$$B_q(n, d) \leq (q - 1)n + 1.$$ 

For $q$-ary constant-weight codes, there is the following Bassalygo’s bound [18], which has been rediscovered many times (for example, in [8]).

Theorem 5 [18]. For any $q$, $n$, $w$, and $d$, we have

$$B_q(n, w, d) \leq \frac{(q - 1)dn}{qw^2 - (q - 1)(2w - d)n}$$

provided that the denominator is positive.

Theorem 6. The numbers $B_q(n, w, d)$ satisfy

$$B_q(n, w, d) \leq \frac{n}{n - w} B_q(n - 1, w, d),$$ \hfill (8)

$$B_q(n, w, d) \leq \frac{n(q - 1)}{w} B_q(n - 1, w - 1, d).$$ \hfill (9)
**Proof.** These inequalities are direct generalizations of the well-known Johnson bounds [19] for binary constant-weight codes. △

Let $P_k(x)$ be the Krawtchouk polynomial, i.e.,

$$P_k(x) = \sum_{i=0}^{k} \binom{x}{i} \binom{n-x}{k-i} (-1)^i (q-1)^{k-i},$$

where

$$\binom{x}{i} = \frac{x(x-1) \ldots (x-i+1)}{i!}$$

for any real $x$.

**Theorem 7** [8]. For any $k$, $k = 1, 2, \ldots, n$, we have

$$B_q(n, d, w) \leq \frac{P_2^2(0) - P_k(d)P_k(0)}{P_k(w) - P_k(d)P_k(0)}$$

provided that the denominator is positive.

**Definition 5.** We say that two $q$-ary codes with the same parameters are equivalent if one can be obtained from the other by a permutation of coordinates of the code and by permutations of alphabet symbols in each position.

### 3. COMBINATORIAL CONSTRUCTION METHODS

Consider a difference matrix $D_q(n)$. It is clear that by adding the corresponding elements of $Q$ to rows and columns of $D_q(n)$, we can reduce this matrix to the standard form where the first row and first column contain only the element 0.

**Theorem 8** [2]. Let $D = D_q(n)$, $n = qk$, be a difference matrix (where $q$ and $k$ are some natural numbers) represented in the standard form. Denote by $D_{-1}$ the matrix obtained from $D$ by deleting the first column. Then $D_{-1}$ is a code matrix of an optimal equidistant $E_q(n,d,N)$-code with parameters

$$n = kq - 1, \quad d = k(q - 1), \quad N = kq.$$  \hfill (10)

Resolvable BIB-designs are equivalent to optimal equidistant codes.

**Theorem 9** [1]. The existence of an RBIB-design $(v, b, r, k, \lambda)$ is equivalent to the existence of an optimal equidistant $E_q(n,d,N)$-code with parameters

$$q = \frac{v}{k}, \quad n = r, \quad d = r - \lambda, \quad N = v.$$  \hfill (11)

A large class of optimal equidistant codes was constructed in [2] (this shows the existence of all RBIB-designs $(v, b, r, k, \lambda)$ for which $v$ and $k$ are arbitrary powers of the same prime number).

**Theorem 10** [2]. Let $g = p^s$, $s = 1, 2, \ldots$, where $p$ is a prime number. Then for arbitrary natural coprime numbers $h$ and $\ell$ (i.e., $(h, \ell) = 1$) there exists an optimal equidistant code $E_q(n,d,N)$ with parameters

$$q = g^h, \quad n = \frac{g^{h+\ell} - 1}{g - 1}, \quad d = g^{\ell} g^h - 1, \quad N = g^{h+\ell}.$$  \hfill (12)

The following simple construction gives all codes with parameters (10) for $k = 2$.  

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Theorem 11. Optimal equidistant codes with parameters (10) for \( k = 2 \), i.e., codes with parameters

\[
n = 2q - 1, \quad d = 2(q - 1), \quad N = 2q,
\]

exist for all natural \( q \geq 2 \).

Proof. Let \( q \geq 2 \) be any natural number. For a vector \( a = (a_1, a_2, \ldots, a_{n-1}, a_n) \) over an alphabet \( Q \) of size \( q = |Q| \), denote by \( a^{(i)} \) the vector obtained from \( a \) by the right cyclic shift by \( i \) positions. Let \( a = a(q) \) be the following vector of length \( 2q - 1 \):

\[
a = (0, 1, 2, \ldots, q - 2, q - 1, q - 1, q - 2, \ldots, 2, 1).
\]

We claim that the matrix

\[
V = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
\vdots \\
a^{(0)} \\
a^{(1)} \\
a^{(2)} \\
\vdots \\
a^{(2q-2)}
\end{pmatrix}
\]

is a matrix of codewords of an optimal equidistant \( q \)-ary code \( E_q(n, d, N) \) with parameters (10) for \( k = 2 \). Only the minimum distance \( d = 2q - 2 \) should be explained. Since all nonzero codewords are formed by cyclic shifts of one and the same codeword, it suffices to prove that for each \( i, j = 1, 2, \ldots, 2q - 2 \), we have the distance \( d(a, a^{(i)}) = 2q - 2 \). Divide the vector \( a \) into three parts:

\[
a = (a^{(0)} | a^{(1)} | a^{(2)}),
\]

where \( a^{(0)} = 0, a^{(1)} = (1, 2, \ldots, q - 1), \) and \( a^{(2)} = (q - 1, q - 2, \ldots, 2, 1) \). Assume that \( a^{(i)} \), where \( i = 1, 2, \ldots, 2q - 2 \), has one common symbol, say \( j \), with the vector \( a^{(1)} \). Taking into account the structure of the vector \( a \), the symbol \( j \) is easily computed, namely, \( j = i/2 \); therefore, \( i \) should be even. Since

\[
a^{(i)} = (i, i - 1, \ldots, j, j - 1, \ldots, 1, 0, 1, 2, \ldots, q - 1, q - 1, \ldots, i + 1),
\]

the codewords \( a \) and \( a^{(i)} \) may coincide in only one position of the vector \( a^{(1)} \). Now assume that the codewords \( a \) and \( a^{(i)} \) coincide in one symbol, say \( s \), of the vector \( a^{(2)} \). This symbol is also easily computed, namely, \( s = q - (i + 1)/2 \); i.e., \( i \) should be odd. Using the same arguments, we conclude that this coincidence is possible in only one element. Therefore, the codewords \( a \) and \( a^{(i)} \) for even \( i \) coincide in one symbol \( j = i/2 \) at the \( j \)th position of \( a^{(1)} \), and for odd \( i \) coincide in one symbol \( s = q - (i + 1)/2 \) at the \( (q - s) \)th position of \( a^{(2)} \). \( \triangle \)

According to Theorem 9, optimal equidistant codes with parameters (10) can be constructed from the corresponding RBIB-designs for the following several values of \( k \): \( k = 3, \ldots, 8 \).

Corollary 1. Optimal equidistant \( E_q(n, d, N) \)-codes with parameters (10) exist for the following values of \( q \) and \( k \):

1. \( k = 3, q \geq 3 \) is any natural number;
2. \( k = 4, q \geq 2 \) is any natural number;
3. \( k = 5, q \geq 4 \) is any natural number with the following possible exceptions:

   \[
   q = 27, 32, 38, 39;
   \]
4. \( k = 6, q \geq 2 \) is any natural number;
(5) \( k = 7, q \geq 3 \) is any natural number with the following possible exceptions:
\[ q = 12, 17, 18, 19, 25, 26, 27, 30, 33, 34, 37, 38, 41, 59, 60, 61, 62, 66; \]

(6) \( k = 8, q \geq 2 \) is any natural number with the following possible exceptions:
\[ q = 3, 20, 21, 24, 28, 30, 39, 42, 55, 69, 70, 93, 183, 186; \]

(7) \( k = q - 1, q \geq 3 \), where both \( k \) and \( q \) are prime powers.

Proof. Cases (1)–(6) follow from the existence of RBIB-designs \((v, b, r, k, \lambda)\) for \( \lambda = k - 1 \) according to Theorem 9. Indeed, by Theorem 9 the existence of an RBIB-design \((v, b, r, k, \lambda = k - 1)\) implies the existence of an \( E_q(n, d, N) \)-code with parameters
\[ n = r = \frac{v - 1}{k - 1} \lambda = v - 1 = qk - 1, \quad d = r - \lambda = k(q - 1), \quad N = qk, \]
where we denoted \( v = qk \) and used formula (5). Therefore, we obtain codes with parameters (10) for the same value of \( k \).

Hanani [21] proved the existence of all RBIB 2-designs \((v, 3, 2), v \geq 9\), which gives statement (1) for \( v \equiv 0 \pmod{3} \).

By efforts of many authors (see [14,22] and references therein), the existence of RBIB 2-designs \((v, k, \lambda = k - 1)\) was proved for \( k = 4, 5, 6, 7, 8 \) (with possible exceptions listed above), which leads to codes given in statements (2)–(6), respectively.

(7) Difference matrices \( D = D_q(n) \) of order \( n = q(q - 1) \) for the cases where both \( q \) and \( q - 1 \) are prime powers are constructed in [23] (see also [13, Section 3.17]). According to Theorem 8, we obtain codes with parameters (10) for \( k = q - 1 \). \( \triangle \)

Corollary 2. Optimal equidistant \( q \)-ary codes with parameters
\[ q = 2\ell + 1, \quad n = 5\ell + 2, \quad d = 5\ell, \quad N = 10\ell + 5 \] (14)
exist for all natural \( \ell \geq 2 \) with the following possible exceptions:
\[ \ell = 4, 11, 13, 19, 21, 23, 29, 31, 33, 34, 39. \]

Proof. According to Theorem 9, codes with parameters (14) correspond to RBIB 2-designs \((v, k = 5, \lambda = 2)\), whose existence is proved for almost all \( \ell \geq 2 \) (see [14, Section 11.7.4]). \( \triangle \)

A number of infinite families of optimal equidistant codes are obtained from the existence of resolvable Steiner systems \( S(v, k, 2) \), i.e., RBIB 2-designs \((v, k, \lambda)\) with \( \lambda = 1 \). First we present codes obtained from RBIB-designs with a fixed parameter \( k \), \( k = 3, 4, 5, 8 \).

Corollary 3. Let \( \ell \geq 1 \) be any natural number. An optimal equidistant code \( E_q(n, d, N) \) with parameters
\[ q = (k - 1)\ell + 1, \quad n = k\ell + 1, \quad d = k\ell, \quad N = k(k - 1)\ell + k \] (15)
exists for the following values of \( k \) and \( \ell \):
(1) \( k = 3, \ell \geq 1 \) is any natural number;
(2) \( k = 4, \ell \geq 1 \) is any natural number;
(3) \( k = 5, \ell \geq 1 \) is any natural number with the following possible exceptions:
\[ \ell = 2, 17, 23, 32; \]
(4) \( k = 8, \ell \geq 1 \) is any natural number with the following possible exceptions:

\[
\ell = 3, 11, 13, 20, 22, 23, 25, 26, 27, 28, 31, 38, 43, 47, 52, 53, 58, 59, 61, 67, 69, 76,
79, 93, 102, 103, 111, 112, 115, 123, 124, 125, 133, 134, 139, 140, 43, 174, 182,
191, 192, 195, 197, 199, 203, 208, 209, 211, 213, 218, 220, 223, 224, 227, 229,
\]

Proof. Statement (1) follows from the result of Ray-Chaudhuri and Wilson [24], who proved the existence of Kirkman triple systems, i.e., resolvable Steiner triple systems \( S(v,3,2) \), for any \( v \equiv 3 \pmod{6} \).

Statement (2) corresponds to RBIB 2-designs \( (v,4,1) \), which exist for all values \( v \equiv 4 \pmod{12} \) (see [25]).

Codes of statement (3) correspond to RBIB 2-designs \( (v,5,1) \) for \( v \equiv 5 \pmod{20} \) (see [22] and references therein).

RBIB 2-designs \( (v,8,1) \) [26] for \( v \equiv 8 \pmod{56} \) imply statement (4). \( \triangle \)

Here are three more families of optimal codes guaranteed by Theorem 9 from RBIB 2-designs with growing \( k \).

Corollary 4. There exist optimal equidistant \( E_q(n,d,N) \)-codes with the following parameters:

\[
q = \ell^2 - \ell + 1, \quad n = \ell^2, \quad d = \ell^2 - 1, \quad N = \ell^3 + 1, \quad (16)
\]

where \( \ell \) is any prime power;

\[
q = 2^\ell - 1, \quad n = 2^\ell + 1, \quad d = 2^\ell, \quad N = 2^{\ell-1}(2^\ell - 1), \quad (17)
\]

where \( \ell \geq 2 \) is any natural number;

\[
q = \ell^2 + 1, \quad n = \ell^2 + \ell + 1, \quad d = \ell^2 + \ell, \quad N = \ell^3 + \ell^2 + \ell + 1, \quad (18)
\]

where \( \ell \) is any prime power.

Proof. RBIB 2-designs \( (\ell^3 + 1, \ell + 1, 1) \) [27], where \( \ell \) is any prime power, lead to codes (16) by Theorem 9.

RBIB 2-designs \( (2^\ell - 1, 2^{\ell-1}, 1) \) [28], where \( \ell \) is any natural number, imply codes (17).

RBIB 2-designs \( (\ell^3 + \ell^2 + \ell + 1, \ell + 1, 1) \) [29], where \( \ell \) is any primer power, give codes (18). \( \triangle \)

Some more particular cases, as well as some construction methods for RBIB-designs, can be found in [14,22,30,31].

Now we give two general constructions of equidistant codes, which are natural generalizations of the constructions of optimal equidistant codes presented in [2].

Definition 6. For a given alphabet \( Q = \{0,1,2,\ldots,q-1\} \), denote by \( Q^{(k)} \), \( k = 1,2,\ldots \), the shift of all its elements by \( (k-1)q \):

\[
Q^{(k)} = (k-1)q + Q = \{(k-1)q,(k-1)q+1,(k-1)q+2,\ldots,(k-1)q+q-1\};
\]

i.e., here we use addition in the ring of integers \( \mathbb{Z} \).

Construction 1. Assume that we have two equidistant codes: a code \( E_{q_1}(n_1,d_1,N_1) \) over an alphabet \( Q_1 = \{0,1,\ldots,q_1-1\} \) and a code \( E_{q_2}(n_2,d_2,N_2) \) over \( Q_2 = \{0,1,\ldots,q_2-1\} \). Let each column of the matrix \( V_2 \) (formed by all codewords of \( E_{q_2} \)) contain each element of \( Q_2 \) the same
number of times \( M_2 = N_2/q_2 \) (this means that this code \( E_{q_2} \) is optimal). Let codewords of \( E_{q_1} \) be enumerated:

\[
E_{q_1} = \{x_1, x_2, \ldots, x_{N_1}\}.
\]

For each codeword \( x_s = (x_{s,1}, x_{s,2}, \ldots, x_{s,n_1}) \), \( s = 1, 2, \ldots, N_1 \), and for any integer \( i \geq 1 \), define a new vector \( x_s^{(i)} \) over the alphabet \( Q_1^{(i)} \):

\[
x_s^{(i)} = (x_{s,1} + (i - 1)q_1, x_{s,2} + (i - 1)q_1, \ldots, x_{s,n_1} + (i - 1)q_1),
\]

where the addition is in the ring \( \mathbb{Z} \). Enumerate codewords \( y \in E_{q_2} \) according to their values in the first position:

\[
E_{q_2} = \{y_{0,1}, y_{0,2}, \ldots, y_{0,M_2}, y_{1,1}, y_{1,2}, \ldots, y_{1,M_2}, \ldots, y_{q_2-1,1}, y_{q_2-1,2}, \ldots, y_{q_2-1,M_2}\},
\]

where each codeword \( y_{s,j} \) has element \( s \in Q_2 \) in the first position. Denote by \( y_{s,j}^{(-1)} \) the vector obtained from \( y_{s,j} \) by deleting the first position (which, as we know, contains the element \( s \)). Let \( M = \min\{N_1, M_2\} \). For any integer \( k, 1 \leq k \leq q_2 \), define a set of vectors

\[
E(k) = \bigcup_{i=1}^{k} \left\{ \bigcup_{j=0}^{M-1} (x_s^{(i)} | y_{s,j}^{(-1)}) \right\}.
\]

This set is a new code of length \( n_1 + n_2 - 1 \) over the alphabet of size \( q = \max(2q_1, q_2) \). The following statement provides a sufficient condition for this code to become equidistant.

**Theorem 12.** Let \( E_{q_1}(n_1, d_1, N_1) \) and \( E_{q_2}(n_2, d_2, N_2) \) be two equidistant codes such that \( d_1 = n_1 - 1 \). Let the code \( E_{q_2} \) be optimal, i.e., let each column of the code matrix \( V_2 \) of \( E_{q_2} \) contain each element of \( Q_2 \) the same number of times. Then for any \( k, 1 \leq k \leq q_2 \), Construction 1 gives an equidistant code \( E_q(n, d, N) \) with parameters

\[
q = \max(kq_1, q_2), \quad n = n_1 + n_2 - 1, \quad d = d_1 + d_2, \quad N = kM,
\]

where \( M = \min\{N_1, N_2/q_2\} \).

**Proof.** The only nontrivial value is the minimum distance \( d \) of the new code. Consider two different codewords \( c = (x_j^{(i)} | y_{i,j}^{(-1)}) \) and \( c' = (x_s^{(i)} | y_{s,j}^{(-1)}) \). We need to consider three cases.

1. Let \( j \neq s \) and \( i = \ell \). Since, by the construction, the vectors \( x_j^{(i)} \) and \( x_s^{(i)} \) (respectively, \( y_{i,j}^{(-1)} \) and \( y_{i,s}^{(-1)} \)) are different codewords of \( E_{q_1}(n_1, d_1, N_1) \) (respectively, \( E_{q_2}(n_2, d_2, N_2) \)), for any \( i \), \( i = 0, 1, \ldots, k - 1 \), we obtain

\[
d(c, c') = d(x_j^{(i)}, x_s^{(i)}) + d(y_{i,j}^{(-1)}, y_{i,s}^{(-1)}) = d_1 + d_2.
\]

2. Now assume that \( j \neq s \) and \( i \neq \ell \). Since \( i \neq \ell \) and the vectors \( x_j^{(i)} \) and \( x_s^{(\ell)} \) are over disjoint alphabets, we have \( d(x_j^{(i)}, x_s^{(\ell)}) = n_1 \). On the other hand, the vectors \( y_{i,j}^{(-1)} \) and \( y_{i,s}^{(-1)} \) are at distance \( d(y_{i,j}^{(-1)}, y_{i,s}^{(-1)}) = d_2 - 1 \) (recall that the first positions of the initial vectors \( y_{i,j} \) and \( y_{i,s} \), where they also differ, are deleted). This results in

\[
d(c, c') = n_1 + (d_2 - 1) = d_1 + d_2.
\]

3. Now let \( j = s \). Since the codewords \( c \) and \( c' \) are different, we conclude that \( \ell \neq i \). Hence, the vectors \( y_{i,j}^{(-1)} \) and \( y_{i,s}^{(-1)} \) are also different, and the vectors \( x_j^{(i)} \) and \( x_s^{(\ell)} \) are over different (disjoint) alphabets. Therefore, we have \( d(c, c') = d_1 + d_2 \), which completes the proof. \( \triangle \)
Construction 2. Let us have an equidistant code $E_{q_1}(n_1, d_1, N_1)$ over an alphabet $Q_1$ and a difference matrix $D = D_{q_2}(m_2, n_2)$ over an alphabet $Q_2$. According to Lemma 2, the rows of $D$ form an equidistant code with minimum distance $d_2 = n_2(q_2 - 1)/q_2$. Enumerate the codewords of $E_{q_1}$:

$$E_{q_1} = \{x_1, x_2, \ldots, x_{N_1}\},$$

and also the rows of $D$:

$$D = \{y_1, y_2, \ldots, y_{m_2}\}.$$

Assume that $Q_2$ is an additive abelian group. Let $M = \min(N_1, m_2)$. Define the following set of vectors:

$$E = \bigcup_{a=0}^{q_2-1} \{(x_j | y_j + a) : j = 1, 2, \ldots, M\},$$

where $a$ denotes a vector with identical components, $a = (a, a, \ldots, a)$, and the addition operation is in the group $Q_2$.

Theorem 13. Let $E_{q_1}(n_1, d_1, N_1)$ be an equidistant code, and let $D = D_{q_2}(m_2, n_2)$ be a difference matrix. Let $d_1 = n_2/q_2$. Then the set $E$ defined above is an equidistant code $E_q(n, d, N)$ with parameters

$$q = \max(q_1, q_2), \quad n = n_1 + n_2, \quad d = n_2, \quad N = q_2 M,$$

where $M = \min(N_1, m_2)$.

Proof. Again, we only have to check the distance between codewords of the new code. Consider any two different codewords $c = (x_j | y_j + a)$ and $c' = (x_s | y_s + a')$.

First assume that $j \neq s$. Then we have

$$d(c, c') = d(x_j, x_s) + d(y_j + a, y_s + a').$$

By the definition of the code $E_{q_1}(n_1, d_1, N_1)$, we have $d(x_j, x_s) = d_1$, and by Lemma 2 we conclude that $d(y_j + a, y_s + a') = n_2(q_2 - 1)/q_2$. Therefore, we have

$$d(c, c') = d_1 + n_2(q_2 - 1)/q_2 = n_2,$$

since $d_1 = n_2/q_2$ by the condition of the theorem.

Let $j = s$. In this case the condition $a \neq a'$ must be valid. Again we use Lemma 2; according to it, we have $d(c, c') = n_2$. Therefore, $d(c, c') = n_2$ for any two different codewords $c$ and $c'$. △

4. EQUIDISTANT CODES WITH DISTANCES $d = 3$ AND $d = 4$

Maximal equidistant codes with distances $d = 3$ and $d = 4$ are completely classified by the following two theorems.

Theorem 14. For $d = 3$ we have

$$B_q(n, 3) = \begin{cases} 
2 & \text{if } q = 2 \text{ and } n \geq 3, \\
q & \text{if } q \geq 2 \text{ and } n = 3, \\
\max\{9, q\} & \text{if } q \geq 3 \text{ and } n \geq 4.
\end{cases}$$

Theorem 14 is equivalent to the following statement, which is more convenient to prove:

$$B_q(n, 3) = \begin{cases} 
2 & \text{if } q = 2 \text{ and } n \geq 3, \\
q & \text{if } q \geq 2 \text{ and } n = 3, \\
9 & \text{if } 3 \leq q \leq 9 \text{ and } n \geq 4, \\
q & \text{if } q \geq 10 \text{ and } n \geq 4.
\end{cases}$$
Proof. We assume that the maximal equidistant code $C$ of size $B_q(n,3)$ to be found always contains the all-zero codeword, which we denote by $c_1 = (0,0,\ldots,0)$.

The case $q = 2$ and $n \geq 3$. Since one of the codewords is the all-zero codeword, all other codewords are of weight 3, which proves this case.

The case $q \geq 2$ and $n = 3$. The trivial equidistant $E_q(3,3,q)$-code is optimal according to Theorem 1.

The case $3 \leq q \leq 9$ and $n \geq 4$. The optimal equidistant code $E_q(n,3,n)$ (Theorem 9) is well known [32] (see also [1,2]). Hence we should only prove that in the interval $3 \leq q \leq 9$, the maximal code $E_q(n,3,N)$ is of size $N = 9$.

Let us show that no equidistant $E_q(n,3,N)$-code $C$ of size $N \geq 10$ exists. First note that any two nonzero codewords $c_i$ and $c_j$ of $C$ intersect in either two or three positions. Consider three subcases, depending on the size of the support of $C$.

$|\text{supp}(C)| \geq 5$. Let $c_2$, $c_3$, and $c_4$ be codewords of $C$ such that

$$|\text{supp}(c_2) \cup \text{supp}(c_3) \cup \text{supp}(c_4)| = 5.$$  

Up to permutations of nonzero elements of the alphabet, these codewords can only be of the form

$$c_2 = (11100\ldots),$$  
$$c_3 = (12011\ldots),$$  
$$c_4 = (13000\ldots).$$

Hence, any other codeword (up to permutations of symbols of the alphabet and permutations of the last $n - 5$ columns) must be of the form

$$c = (140001\ldots).$$

Hence we conclude that such a code $C$ is of cardinality at most $q$; i.e., we get a contradiction.

$|\text{supp}(C)| = 4$. Since the cardinality of the code is $N \geq 10$, there exist at least four codewords with the zero element in one and the same position. Indeed, all nine nonzero codewords of the code have nine zeros, which must be distributed in four positions. Hence, for one position, say the first, the element 0 (taking into account the all-zero codeword $c_1$) must appear exactly four times. Thus, up to permutations of symbols of the alphabet, four codewords of $C$ are known:

$$c_1 = (00000\ldots),$$  
$$c_2 = (01111\ldots),$$  
$$c_3 = (02222\ldots),$$  
$$c_4 = (03333\ldots).$$

Now there are only two possible cases: either the next codeword (up to permutations of the symbols $4,5,\ldots,q-1$ of the alphabet) is of the form

$$c_5 = (04444\ldots)$$

and in this case the cardinality of the code is exactly $q$, or $q = 4$ and we cannot add any more codeword. In both cases we obtain a contradiction.

$|\text{supp}(C)| = 3$. This case corresponds to the trivial $E_q(n,3,q)$-code $C$ of cardinality $q$; i.e., the code cannot have cardinality $N \geq 10$.

Hence, it is proved that if $n \geq 4$ and $3 \leq q \leq 9$, then $B_q(n,3) = 9$.

The case $q \geq 10$ and $n \geq 4$. If $q > 9$, then $B_q(n,3) = q$. Both nonequivalent maximal codes $C$ of size $q$—one with support size $|\text{supp}(C)| = q + 1$, and the other (optimal) with support size $|\text{supp}(C)| = 3$—were considered in the previous cases. △
Theorem 15. For $d = 4$ we have

$$B_q(n, 4) = \begin{cases} 
q & \text{if } q \geq 2 \text{ and } n = 4, \\
2 & \text{if } q = 2 \text{ and } n \leq 5, \\
4 & \text{if } q = 2 \text{ and } n = 6, \\
n + 1 & \text{if } q = 3 \text{ and } n = 5, 6, \\
\max\{8, \lfloor n/2 \rfloor\} & \text{if } q \leq 3 \text{ and } n \geq 7, \\
\max\{16, \lfloor n/2 \rfloor, q\} & \text{if } q \geq 4 \text{ and } n \geq 5.
\end{cases}$$

Again we formulate Theorem 15 in a more convenient form:

$$B_q(n, 4) = \begin{cases} 
q & \text{if } q \geq 2 \text{ and } n = 4, \\
2 & \text{if } q = 2 \text{ and } n \leq 5, \\
4 & \text{if } q = 2 \text{ and } n = 6, \\
n + 1 & \text{if } q = 3 \text{ and } n = 5, 6, \\
\max\{8, \lfloor n/2 \rfloor\} & \text{if } q \leq 3 \text{ and } n \geq 7, \\
\max\{16, \lfloor n/2 \rfloor\} & \text{if } q \geq 4 \text{ and } n \geq 33.
\end{cases}$$

Proof. Let $C$ be a maximal $E_q(n, 4, N)$-code containing the all-zero codeword.

The case $q \geq 2$ and $n = 4$. The trivial $E_q(4, 4, q)$-code is optimal (Theorem 1) and contains (up to permutations of symbols of the alphabet) all codewords $c_i$ of the form $c_i = (i\ldots i)$, where $i \in Q$.

The case $q = 2$ and $n \leq 5$. The maximal code obviously contains two codewords.

The case $q = 2$ and $n = 6$. The optimal $E_2(6, 4, 4)$-code $C$ consists of two repetitions of the Hadamard $E_2(3, 2, 4)$-code

$$\{(000), (011), (101), (110)\}.$$  

The case $q \leq 3$ and $n = 5, 6$. For $n = 5$, the optimal (unique up to equivalence) $E_3(5, 4, 6)$-code $C$ exists by Theorem 11 and consists of the following codewords:

$$c_1 = (00000),$$
$$c_2 = (11110),$$
$$c_3 = (22102),$$
$$c_4 = (10222),$$
$$c_5 = (21021),$$
$$c_6 = (02211).$$

For $n = 6$ there also exists a unique maximal (but not optimal since symbols in each position occur with different frequencies) $E_3(6, 4, 7)$-code $C$, containing the codewords

$$c_1 = (000000),$$
$$c_2 = (111100),$$
$$c_3 = (110011),$$
$$c_4 = (001111),$$
$$c_5 = (221010),$$
$$c_6 = (102210),$$
$$c_7 = (101022).$$
The maximality of this code is easy to prove. Since an $E_3(6,4,7)$-code is unique, for the maximality it suffices to prove that this code is impasse. This fact is easy to check (Lemma 3 significantly reduces the number of cases).

The case $q \leq 3$ and $7 \leq n \leq 17$. For this case we state that the maximal code $E_3(n,4,N)$ is the binary optimal Hadamard code $E_2(7,4,8)$ with $n - 7$ zero positions added. For lengths $n = 16$ and $n = 17$ we also have one more trivial maximal code $E_2(n,4,8)$, containing the codewords

$$c_1 = (0000000000\ldots),$$
$$c_2 = (1111000000\ldots),$$
$$c_3 = (1100110000\ldots),$$
$$c_4 = (1100001100\ldots),$$
$$c_5 = (1100000011\ldots),$$

(19)

with an added trivial column if $n = 17$. Hence, for $n \geq 7$, none of the ternary codes $E_3(n,4,N)$ is of size $N \geq 8$. The proof of this fact is based on considerations of the support size of such codes. Indeed, without loss of generality, the code $C$ contains the codewords

$$c_1 = (0000000),$$
$$c_2 = (1111000).$$

Note that the first four positions of any other codeword $c_3$ of $C$, up to permutations of positions, are of the form $c_3 = (2222\ldots)$, $c_3 = (2210\ldots)$, or $c_3 = (1100\ldots)$. The corresponding impasse codes are easy to enumerate for all the three cases.

The case $c_3 = (2222000)$. It is easy to see that the code $C = \{c_1, c_2, c_3\}$ is impasse.

The case $c_3 = (2210abc)$. Besides the maximal code $E_3(6,4,7)$ given above, for this case there is only one (up to equivalence) impasse code with support size 7:

$$c_1 = (0000000),$$
$$c_2 = (1111000),$$
$$c_3 = (2210100),$$
$$c_4 = (2120010),$$
$$c_5 = (1220001)$$

and three nonequivalent impasse codes with support size 6:

$$c_1 = (000000), \quad c_1 = (000000), \quad c_1 = (000000),$$
$$c_2 = (111100), \quad c_2 = (111100), \quad c_2 = (111100),$$
$$c_3 = (221010), \quad c_3 = (221010), \quad c_3 = (221010),$$
$$c_4 = (110011), \quad c_4 = (110011), \quad c_4 = (110011),$$
$$c_5 = (210220), \quad c_5 = (210220), \quad c_5 = (220101),$$
$$c_6 = (102210), \quad c_6 = (011022), \quad c_6 = (001111).$$

The case $c_3 = (1100110)$. The code with the codewords $c_1, c_2, c_3$ can be extended to some of the above-given codes (with support size 6). In this case, of course, we obtain binary maximal codes, namely, the binary Hadamard $E_2(7,4,8)$-code (with support size 7) and the trivial $E_2(n,4,\lfloor n/2 \rfloor)$-code with support size $2\lfloor n/2 \rfloor$. 

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The case \( q \leq 3 \) and \( n \geq 17 \). In this case we have one trivial maximal \( E_2(n, 4, \lfloor n/2 \rfloor) \)-code, containing codewords (19). Indeed, as is shown in the previous case, none of the ternary codes \( E_3(n, 4, N) \) is of size \( N \geq 8 \).

The case \( q \leq 16 \) and \( 5 \leq n \leq 33 \). We claim that for this case the well-known optimal \( E_4(5, 4, 16) \)-code (Theorem 10) \([32]\) (see also \([1, 2]\)) gives a maximal code for any \( q \leq 16 \) and \( n \leq 33 \). Let us show that there are no other nontrivial cases. We consider several subcases according to the support size of \( C \).

\( |\text{supp}(C)| = 4 \). It is clear that for the case of support size 4, the trivial optimal \( E_q(5, 4, q) \)-code is unique up to equivalence.

\( |\text{supp}(C)| = 5 \). Consider codes with support size 5. If \( q = 4 \), then the above-mentioned \( E_4(5, 4, 16) \)-code is optimal according to the Plotkin bound (Theorem 3).

Hence we should only consider the case \( q \geq 5 \). Let us show that the cardinality of the code \( C \) in any case is at most 16. It is clear that \( C \) contains the vectors \( c_1 = (00000) \) and \( c_2 = (11110) \). Since the distance between codewords is 4, any other codeword which is different from \((a \ a \ a \ a \ 0)\), where \( a \neq 0, 1 \), contains in the first four positions exactly one element 0 and one element 1. Moreover, for the case where the support size is 5 (and only for this case), positions of these two elements uniquely determine the codeword. But in these four positions there are exactly 12 different choices for two positions with the elements 0 and 1. Hence, only two different situations are possible.

If \( q = 4 \), then besides these twelve codewords (defined by two positions with the elements 0 and 1), only two codewords \( c_3 = (22220) \) and \( c_4 = (33330) \) can be added. Thus, we obtain an optimal \( E_4(5, 4, 16) \)-code.

If \( q \geq 5 \), then the cardinality of a maximal code \( C \) is at most 16. Indeed, there are exactly twelve codewords having two positions with elements 0 and 1 in the first four coordinate positions. Let \( c = c(i_1, i_2) \) be one of these codewords with 0 in position \( i_1 \) and 1 in position \( i_2 \). We claim that for \( q > 4 \) it is possible to add (besides these twelve codewords) only two codewords, say \( c_3 = (a a a a 0) \) and \( c_4 = (b b b b 0) \). Since the length of the code is \( n = 5 \), any such codeword \( c \) must also contain (besides 0 and 1) elements \( a \) and \( b \) in the first four coordinate positions. This is the only possibility for the equalities \( d(c_3, c) = 4 \) and \( d(c_4, c) = 4 \) to hold for all codewords \( c \) of the code \( C \). Hence, it is impossible (irrespective of the alphabet size) to add to the existing sixteen codewords (four codewords \( c_i, i = 1, 2, 3, 4 \), and twelve codewords of the type \( c = c(i_1, i_2) \) with elements 0 and 1) any more codeword that would be at distance 4 from all codewords of \( C \).

It only remains to prove that the obtained code is an \( E_4(5, 4, 16) \)-code over the alphabet \( \{0, 1, a, b\} \). Recall that \( \xi^{(j)}_\gamma \) is the multiplicity of occurrence of an element \( \gamma \) in the \( j \)-th column of the code matrix \( V \). Since the first four columns of \( V \) contain only elements 0, 1, \( a \), and \( b \), we have \( \xi^{(j)}_\gamma = 4, j = 1, 2, 3, 4 \). Let the fifth column contain only elements of the set \( \{0, 1, a, b\} \). It is clear that any such element occurs at most four times (indeed, in the first four positions there can be only four codewords at distance 4 from each other). This gives \( \xi^{(5)}_\gamma = 4 \) for any \( \gamma \in \{0, 1, a, b\} \). Then for the parameter \( \Gamma \) of the obtained code (Lemma 3) we have

\[
\Gamma = 16 \cdot 15 = 5 \cdot 4 \cdot (4 \cdot 3) = 240.
\]

Now assume that in the codeword matrix \( V \) there exists a column containing five or more different elements of the alphabet. As is proved above, a code with sixteen codewords contains in the first four coordinates exactly four different elements of the alphabet. Assume that the fifth column contains five different elements. In the best case (where three symbols occur four times each, one symbol occurs three times, and one more symbol occurs once) the contribution \( \Gamma_5 \) of this column to \( \Gamma \) can be upper bounded as

\[
\Gamma_5 \leq 3 \cdot (4 \cdot 3) + 3 \cdot 2 = 42.
\]
Hence,
\[ 240 = \Gamma \leq 4 \cdot (4 \cdot (4 \cdot 3)) + 42 = 234, \]
which contradicts Lemma 3.

\[ |\text{supp}(C)| \geq 6. \] For this case, as above, it is easy to enumerate all impasse codes.

Let us have a maximal \( E_q(n,4,N) \)-code \( C \) over the alphabet of size \( q \geq 4 \). Without loss of generality, \( C \) contains two codewords, \( c_1 = (00000 \ldots) \) and \( c_2 = (11110 \ldots) \). Since \( C \) contains the codeword \( c_2 \), all other possible codewords \( c = (c_{i,1} c_{i,2} c_{i,3} c_{i,4} \ldots) \) having four nonzero elements \( c_{i,s} \neq 0, s = 1,2,3,4, \) in the first four positions can be reduced by permutations of elements of the alphabet to the form where all \( c_{i,s} \) are the same. Let \( c_3 = (22220 \ldots) \) and \( c_4 = (33330 \ldots) \) be such possible codewords. Also, there are codewords with three nonzero elements in the first four positions containing elements 0 and 1 in these positions. Let \( c_{(i_1,i_2)} \) be one of these codewords with elements 0 and 1 in positions \( i_1 \) and \( i_2 \), respectively, where \( i_1, i_2 \in \{1,2,3,4\} \) and \( i_1 \neq i_2 \). Finally, codewords containing two elements 1 and two elements 0 in the first four positions are also possible.

There are three subcases to consider.

\textit{The case} \( c_3, c_4 \in C \). This case is easy to resolve. Indeed, since \( C \) contains the codewords
\[
\begin{align*}
c_1 &= (00000 \ldots), \\
c_2 &= (11110 \ldots), \\
c_3 &= (22220 \ldots), \\
c_4 &= (33330 \ldots),
\end{align*}
\]
there are exactly twelve codewords of the form \( c_{(i_1,i_2)} \) that contain four different elements 0, 1, 2, 3 in the first four positions. Let us enumerate these twelve codewords with the numbers of positions of elements 0 and 1, which uniquely determine these codewords. Let \( c_{(i_1,i_2)} \) be one of these codewords with elements 0 and 1 in positions \( i_1 \) and \( i_2 \), respectively, where \( i_1, i_2 \in \{1,2,3,4\} \) and \( i_1 \neq i_2 \).

Let us show that all codewords \( c_{(i_1,i_2)} \) can be uniquely extended, up to permutations of the symbols 2 and 3, to complete codewords of weight 4. Indeed, consider three vectors \( c_{(i_1,i_2)} \) with four unknown elements \( a,b,c,d \in \{2,3\} \):
\[
\begin{align*}
c_{(1,2)} &= (01230 \ldots), \\
c_{(1,3)} &= (0a1b0 \ldots), \\
c_{(1,4)} &= (0c,d10 \ldots).
\end{align*}
\]

First let us check that all these three vectors are at distance 3 from each other, i.e., are of the form
\[
\begin{align*}
c_{(1,2)} &= (01230 \ldots), \\
c_{(1,3)} &= (03120 \ldots), \\
c_{(1,4)} &= (02310 \ldots). \quad (20)
\end{align*}
\]
Furthermore, the fourth nonzero position is determined uniquely; i.e., there exists a position, say \( j^* \), such that these vectors contain the elements 1, 2, and 3 in this position \( j^* \). For example,
\[
\begin{align*}
c_{(1,2)} &= (012301 \ldots), \\
c_{(1,3)} &= (031202 \ldots), \\
c_{(1,4)} &= (023103 \ldots). \quad (21)
\end{align*}
\]
Assume that this is not true. Since each vector contains the elements 2 and 3, the only possible situation up to permutations of these elements is the following:

\[
c(1,2) = (012310\ldots), \\
c(1,3) = (021302\ldots), \\
c(1,4) = (023130\ldots). \\
\]  

(22)

In this case, extending the three vectors \(c(2,i_2)\) becomes questionable. There are two possibilities, which depend on the distance between the first vectors of the triples \(c(1,2)\) and \(c(2,1)\) in the first four position. Let this distance be 2, i.e.,

\[
c(2,1) = (10230\ldots), \\
c(2,3) = (20130\ldots), \\
c(2,4) = (20310\ldots). \\
\]

Then the first two vectors can be extended uniquely:

\[
c(2,1) = (102301\ldots), \\
c(2,3) = (201330\ldots). \\
\]

The third vector \(c(2,4) = (20310\ldots)\) cannot be extended because this vector is at distance 4 from the two vectors \(c(1,2)\) and \(c(1,3)\) of formula (22). Similarly, we rule out the other possible case

\[
c(2,1) = (10320\ldots), \\
c(2,3) = (20130\ldots), \\
c(2,4) = (20310\ldots). \\
\]

Thus, it is proved that the triple of vectors \(c(1,i_2)\) is of the form (20), and after the extension it looks like (21). Therefore, the triple of vectors \(c(2,i_2)\) is as follows:

\[
c(2,1) = (10320\ldots), \\
c(2,3) = (20130\ldots), \\
c(2,4) = (30210\ldots). \\
\]

(23)

Now let us show that for all these triples of vectors, the additional nonzero position is the same. Recall that \(j^*\) is the number of the additional position for the vectors \(c(1,i_2)\). Let \(j^{**}\) be the number of the additional position for the vectors \(c(2,i_2)\). Since \(d(c(1,2), c(2,1)) = 4\), these vectors must coincide in the last \(n - 4\) positions; i.e., \(j^* = j^{**}\). Hence, all the 12 vectors \(c(i_1,i_2)\) contain the fourth nonzero element in one and the same position, i.e., in position 5. Thus, together with the four vectors \(c_i, i = 1, 2, 3, 4\), we obtain an optimal \(E_4(5,4,16)\)-code, i.e., a code with support size 5.

It is proved above that for a code with support size 5, the cardinality of the alphabet has no effect on the cardinality of the maximal code. In other words, a maximal \(E_q(5,4,N)\)-code with support size 5 is nothing else but the optimal \(E_4(5,4,16)\)-code.

Let us try to add one more codeword \(c_{17}\) by increasing the support size. A possible codeword contains elements 0, 1, 2, and 3 in the first four positions and some element, say \(a\), in the sixth position. This is the only possibility for \(c_{17}\) to be at distance 4 from all codewords of the code \(E_4(5,4,16)\).

Let us use Lemma 3. Since the code \(E_4(5,4,16)\) is optimal, the multiplicity of occurrence of elements in the first five columns is the same, namely, \(\xi^{(j)}_7 = 4\), where \(j = 1, 2, 3, 4, 5\) and
γ ∈ {0, 1, 2, 3}. Assuming the possibility of adding the vector \( \mathbf{c}_{17} = (0 \ 1 \ 3 \ 2 \ 0 \ a \ldots) \) to this code, we obtain the following possible multiplicities of elements:

\[
\xi_0^{(1)} = \xi_1^{(2)} = \xi_2^{(3)} = \xi_3^{(4)} = \xi_4^{(5)} = 5.
\]

where the multiplicities of all other elements in the first five columns are equal to 4. For the sixth column, we have \( \xi_0^{(6)} = 16 \) and \( \xi_2^{(6)} = 1 \). According to Lemma 3, for an \( E_q(6, 4, 17) \)-code we have

\[
\Gamma = 2 \cdot (17 \cdot 16) = 544.
\]

On the other hand, using the multiplicity \( \xi_{(j)}^{(j)} \), we obtain

\[
\Gamma \leq 5 \cdot (5 \cdot 4 + 3(4 \cdot 3)) + 16 \cdot 15 = 520,
\]

i.e., a contradiction.

The case \( \mathbf{c}_3 \in C \) but \( \mathbf{c}_4 \notin C \). Note that any other codeword contains each of the elements 0, 1, and 2 exactly once in the first four positions. Let us show that under these conditions the maximal \( E_q(n, 4, N) \)-code has cardinality \( N \leq 15 \). To prove this, it suffices to show that in each of the first four positions, the element 0 occurs at most four times, i.e., that \( \xi_0^{(j)} \leq 4 \), where \( j = 1, 2, 3, 4 \). Under these conditions, the cardinality of the code is at most 15 since there are twelve codewords \( \mathbf{c}(i_1, i_2) \) (with different ordered pairs \( (i_1, i_2) \)) and three codewords \( \mathbf{c}_1 \), \( i = 1, 2, 3 \).

Assume the contrary, i.e., let there exist a position in which the element 0 occurs at least five times. Assume that this is the first position; then the code \( C \) contains the following codewords (up to permutations of positions):

\[
\mathbf{c}_1 = (0 \ 0 \ 0 \ 0 \ 0 \ 0 \ldots),
\]
\[
\mathbf{c}_2 = (1 \ 1 \ 1 \ 1 \ 0 \ 0 \ldots),
\]
\[
\mathbf{c}_3 = (2 \ 2 \ 2 \ 2 \ 0 \ 0 \ldots),
\]
\[
\mathbf{c}(1, 2) = (0 \ 1 \ 2 \ a \ a_1 \ a_2 \ldots),
\]
\[
\mathbf{c}(1, 3) = (0 \ b \ 1 \ 2 \ b_1 \ b_2 \ldots),
\]
\[
\mathbf{c}(1, 4) = (0 \ 2 \ c \ 1 \ c_1 \ c_2 \ldots),
\]
\[
\mathbf{c} = (0 \ 1 \ d \ 2 \ d_1 \ d_2 \ldots).
\]

Here \( a, b, c, d, a_i, b_i, c_i, d_i, \ i = 1, 2 \), are some elements of the alphabet \( Q \). Let us show that for any choice of the elements \( a, b, c, d, a_i, b_i, c_i, d_i \), the corresponding vectors do not form a subcode of an equidistant \( E_q(n, 4, N) \)-code.

First note that \( a, b, c, d \notin \{0, 1, 2\} \). Indeed, if \( a = 2 \), then \( d(\mathbf{c}_3, \mathbf{c}(1, 2)) = 3 \) for any \( a_1 \neq 0 \) since it is obvious that \( a_2 = 0 \). By the same reason, \( a, b, c, d \notin \{0, 1, 2\} \) and \( a, b, c, d \neq 0 \). Therefore, among the elements \( a_1, a_2 \) (respectively, \( b_1, b_2 \) and also \( c_1, c_2 \) and \( d_1, d_2 \)) there is exactly one nonzero element. Furthermore, in the column \( j, \ j = 5, 6 \), all possible nonzero elements among \( a_1, b_1, c_1, d_1 \) (respectively, among \( a_2, b_2, c_2, d_2 \)) are different. Indeed, if \( a_1 = b_1 \), we obtain \( d(\mathbf{c}(1, 2), \mathbf{c}(1, 3)) = 3 \).

Use Lemma 3 again. Assume that there exist elements \( a, b, c, d, a_i, b_i, c_i, d_i, \ i = 1, 2 \), such that all codewords listed above are at distance 4 from each other. Consider the multiplicity matrix \( \Phi \) for the above subcode, writing only known multiplicities \( \xi_{(j)}^{(j)} \) for \( \gamma \in \{0, 1, 2, c\} \) and \( j = 1, \ldots, 6 \):

\[
\Phi = \begin{pmatrix}
5 & 1 & 1 & 1 & 5 & 5 \\
1 & 3 & 2 & 2 & * & * \\
1 & 2 & 2 & 3 & * & * \\
* & * & 2 & * & * & *
\end{pmatrix}.
\]
Here the symbol $*$ denotes an unknown element equal to either 0 or 1, which does not affect the sum $\sum_{j,\gamma} \xi_{\gamma}^{(j)}(\xi_{\gamma}^{(j)} - 1)$; we have also assumed that $c = d$, i.e., $\xi_{c}^{(3)} = 2$ (which only increases the sum).

Thus, according to Lemma 3, the value of $\Gamma$ for this subcode (of cardinality 7) is

$$\Gamma = N(N - 1)(n - d) = 84.$$  

On the other hand, using the multiplicities $\xi_{\gamma}^{(j)}$ in the above matrix $\Phi$, we obtain the following upper bound:

$$\Gamma \leq \sum_{j=1}^{6} \left( \sum_{\gamma \in \{0,1,2,c\}} \xi_{\gamma}^{(j)}(\xi_{\gamma}^{(j)} - 1) \right) = 3 \cdot (5 \cdot 4) + 2 \cdot (3 \cdot 2 + 2) + 3 \cdot 2 = 82,$$

which contradicts Lemma 3.

If we assume that the subcode has support size 7, then in a similar way we obtain

$$\Gamma = N(N - 1)(n - d) = 42 \cdot 3 = 126.$$  

On the other hand, multiplicities of occurrence of elements yield the bound $\Gamma \leq 122$, which is impossible. This completes the proof in this case.

The case $c_3, c_4 \notin C$. As is shown above, under this condition any other codeword except for $c_1 = (00000 \ldots)$ and $c_2 = (11110 \ldots)$ contains in the first four positions at least one element 0 and one element 1. As above, we denote by $c(i_1, i_2)$ such a codeword containing 0 in position $i_1$ and 1 in position $i_2$. Since there are twelve different ordered pairs $(i_1, i_2)$ with $i_1, i_2 \in \{1,2,3,4\}$ and $i_1 \neq i_2$, it suffices to consider the following possible subcode (which we denote by $C_1$) of the maximal equidistant code $E_q(n,4,N)$ with support size 6 or larger:

$$c_1 = (000000 \ldots),$$
$$c_2 = (111100 \ldots),$$
$$c(1,2) = (0122100 \ldots),$$
$$c(1,2)^* = (0133010 \ldots).$$

Let us show that the code $C_1$ can be extended to either an impasse $E_q(6,4,N)$-code $C$ of cardinality $N \leq 15$ or an impasse code $E_q(n,4,q)$ with support size $q + 2$.

For the codeword matrix $V$, the condition $c_3, c_4 \notin C$ means that for any two its (different) columns, any ordered pair of elements occurs as a row at most twice. Then in the matrix $V$ of a possible code $C$ of cardinality 16, all positions of zero elements are known (these are all 15 possible pairs of positions plus the zero codeword); i.e., we have the multiplicity $\xi_{\gamma}^{(j)} = 6$ for $j = 1, \ldots, 6$. Using this, it is easy to prove that $\xi_{\gamma}^{(j)} \leq 5$ for any element $\gamma \neq 0$ and any $j = 1, \ldots, 6$. This upper bounds the value of $\Gamma$ for such a code: $\Gamma \leq 420$, but by Lemma 3 we have $\Gamma = 480$; i.e., such a code does not exist.

Now let $|\text{supp}(C)| \geq 7$. A code with support size 7 can be constructed by extending a code with support size 6. In order not to lose the generality, note that the (nonbinary) code $E_3(6,4,4)$ with support size 6 (which is unique up to equivalence) is uniquely extended to the unique impasse code $E_3(7,4,5)$ with support size 7, mentioned above.

The unique (up to equivalence) quaternary code $E_4(6,4,4)$ (which is neither binary nor ternary) with support size 6 contains the codewords given in (24). Using Lemma 3 and equalities (3) and (4), we extend this code to a code with support size 7. We see that the only possible extension of the
code $C_1$ when the support size increases is obtained when using elements different from 0, 1, 2, and 3, and the corresponding multiplicity matrix is as follows:

$$\Phi = \begin{pmatrix}
4 & 1 & 1 & 1 & 4 & 4 & 4 \\
1 & 4 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0
\end{pmatrix}.$$  

We check that $\Gamma = 5 \cdot (4 \cdot 3) = 60$. Thus, up to permutations of the elements of the alphabet, the vector $c$ added to the code $C_1$ is

$$c = (0 1 4 4 0 0 1 \ldots).$$

Obviously, we arrive at the trivial (maximal for $q \geq 16$) impasse code $E_q(n, 4, q)$ with support size $q + 2$. 

To complete this case, it suffices to mention another trivial impasse $E_q(n, 4, \lfloor n/2 \rfloor)$-code containing only binary codewords of the form (19).

The case $q \geq 16$ and $5 \leq n \leq 33$. In this case, besides the optimal code $E_q(n, 4, 16)$ with support size 5, there are also two trivial maximal $E_q(n, 4, q)$-codes: one with support size 4 (optimal) and the other with support size $q + 2$.

The case $q \geq 16$ and $n \geq 33$. This case is also trivial. Besides the optimal code $E_q(n, 4, q)$ with support size 4 and the maximal code $E_q(n, 4, 16)$ with support size $q + 2$, there is also a maximal code $E_q(n, 4, \lfloor n/2 \rfloor)$ containing only binary codewords of the form (19).

Thus, the maximal code $E_q(n, 4, N)$ for $q \geq 4$ and $n \geq 5$ has cardinality $N = \max\{16, q, \lfloor n/2 \rfloor\}$.

This completes the proof of Theorem 15. $\triangle$

5. COMPUTER SEARCH

5.1. Exhaustive search. Let $C$ be an equidistant $E_q(n, d, N)$-code. Our approach is based on the observation that such an $E_q(n, d, N)$-code $C$ can be shortened to an $E_q(n - 1, d, N')$-code $C'$ of cardinality $N'$. The value $N'$ can be lower bounded by averaging over all $n$ positions, namely,

$$N' \geq \left\lfloor \frac{n^2 - d}{n} \right\rfloor + 1.$$  \hspace{1cm} (25)

Indeed, each of the $N - 1$ nonzero codewords of $C$ contains $n - d$ zeros, which are in some way distributed in $n$ positions, which results in the lower bound (25). Furthermore, one can use the relation between equidistant and equidistant constant-weight codes and the fact that any code $E_q(n, w, d, N)$ contains some $E_q(n - 1, w, d, N')$-code, whose size can be lower bounded according to Theorem 6. Taking also into account the zero codeword, we obtain the bound (25).

Conversely, an $E_q(n, d, N)$-code $C$ can be constructed by extending an already existing $E_q(n - 1, d, N')$-code $C'$. In this way it is possible to construct an equidistant constant-weight code $E_q(n, w, d, N)$ by extending an already existing code $E_q(n - 1, w, d, N')$ or $E_q(n - 1, w - 1, d, N'')$.

Two main problems of algebraic coding theory are well known: construction of codes of the maximum possible cardinality $N$ for given $q$, $n$, and $d$ and classification of all nonequivalent codes with given parameters $q$, $n$, $d$, and $N$.

In our case the first problem reduces to constructing all $E_q(n, d, N)$-codes with $N$ codewords that contain a given code $C'$ as a subcode. This problem can be represented as the maximal clique problem in the graph induced by the set of all $q$-ary vectors of length $n$; to solve it, we use the so-called backtrack search. The search space is only the set of all vectors that are at distance $d$ from every codeword of $C'$. Then only the distance between codewords is under control.
ON THE CONSTRUCTION OF $q$-ARY EQUIDISTANT CODES

$B_q(n,d)$ for $n \leq 10$

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Notes. No index: computer search; $a$: Theorem 9; $b$: Theorem 14; $c$: Theorem 15; $d$: Theorem 10; $e$: Theorem 11; $f$: Corollary 3; $g$: [1].

For the case of $q$-ary codes, the corresponding graph where the search is realized is defined as follows. Vertices are vectors of length $n$ over the alphabet of size $q$, and two vertices are connected by an edge if and only if the Hamming distance between the corresponding vectors is exactly $d$. What we have to find in the graph thus obtained is the quantity $B_q(n,d)$, the size of the largest clique in this graph.

5.2. Constructive search. Since this kind of search is not always applicable, especially when values of the parameters $n$, $q$, and $d$ become too large, we modify this algorithm in a suitable way. The modifications are based on a method for solving the maximum clique problem. A part of this problem is a problem of constructing codes derived from the constructions presented in Section 3. Using platform-independent Java technologies, we created a library of classes and interfaces and a search system (GUI) for them. This software implements the algorithms of all the presented constructions and constructs new codes.

Both softwares—for the general code and for the case of code obtained by one of the above constructions—are included as modules in the computer system [33].

The second problem is to find all inequivalent codes with parameters $q$, $n$, $d$, and $N$. To solve this problem, we use methods similar to those described in [34]. If we have to check whether two codes are equivalent or not, first we convert these codes into graphs and then check these two graphs for isomorphism using such systems as the well-known Nauty system [35] and others [33,36].
All general construction methods for equidistant (and equidistant constant-weight) codes were systematized using classical combinatorial designs such as RBIB-designs, Latin squares, difference matrices, and so on. Using computer search, we have found many new maximal equidistant codes which cannot be obtained by methods based on known combinatorial configurations.

As was already mentioned, most of the new codes are obtained using a computer program based on backtrack search [33,34].

As an example of codes obtained in this way, we present codewords of four nonequivalent $E_6(7, 6, 18)$-codes:

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